Chapter 13

Vectors

Vectors are arrows which have a definite magnitude and direction.

Many interesting mathematical and physical quantities depend upon more than one independent variable. The length z of the hypotenuse of a right triangle, for instance, depends upon the lengths x and y of the two other sides; the dependence is given by the Pythagorean formula $z = \sqrt{x^2 + y^2}$. Similarly, the growth rate of a plant may depend upon the amounts of sunlight, water, and fertilizer it receives; such a dependence relation may be determined experimentally or predicted by a theory.

The calculus of functions of a single variable, which we have been studying since the beginning of this book, is not enough for the study of functions which depend upon several variables—what we require is the *calculus of functions of several variables*. In the final six chapters, we present this general calculus.

In this chapter and the next, we set out the algebraic and geometric preliminaries for the calculus of several variables. This material is thus analogous to Chapter R, but not so elementary. Chapters 15 and 16 are devoted to the differential calculus, and Chapters 17 and 18 to the integral calculus, of functions of several variables.

13.1 Vectors in the Plane

The components of vectors in the plane are ordered pairs.

An ordered pair (x, y) of real numbers has been considered up to now as a point in the plane—that is, as a *geometric* object. We begin this section by giving the number pairs an *algebraic* structure.[1] Next we introduce the notion of a vector. In Section 13.2, we discuss the representation of points in space by triples (x, y, z) of real numbers, and we extend the vector concept to three dimensions. In Section 13.3, we apply the algebra of vectors to the solution of geometric problems.

The authors of this book, and probably many of its readers, were brought up mathematically on the precept that "you cannot add apples to oranges." If we have x apples and y oranges, the number $x + y$ represents the number of

[1] The reader who has studied Section 12.6 on complex numbers will have seen some of this algebraic structure already.

pieces of fruit but confounds the apples with the oranges. By using the ordered pair (x, y) instead of the sum $x + y$, we can keep track of both our apples and our oranges without losing any information in the process. Furthermore, if someone adds to our fruit basket X apples and Y oranges, which we might denote by (X, Y), our total accumulation is now $(x + X, y + Y)$—that is, $x + X$ apples and $y + Y$ oranges. This addition of items kept in separate categories is useful in many contexts.

Addition of Ordered Pairs

If (x_1, y_1) and (x_2, y_2) are ordered pairs of real numbers, the ordered pair $(x_1 + x_2, y_1 + y_2)$ is called their *sum* and is denoted by $(x_1, y_1) + (x_2, y_2)$. Thus, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Example 1 (a) Calculate $(-3,2) + (4,6)$.
(b) Calculate $(1,4) + (1,4) + (1,4)$.
(c) Given pairs (a,b) and (c,d), find (x, y) such that $(a,b) + (x, y) = (c,d)$.

Solution (a) $(-3,2) + (4,6) = (-3 + 4, 2 + 6) = (1,8)$.
(b) We have not yet defined the sum of three ordered pairs, so we take the problem to mean $[(1,4) + (1,4)] + (1,4)$, which is $(2,8) + (1,4) = (3,12)$. Notice that this is $(1 + 1 + 1, 4 + 4 + 4)$, or $(3 \cdot 1, 3 \cdot 4)$.
(c) The equation $(a,b) + (x, y) = (c,d)$ means $(a + x, b + y) = (c,d)$. Since two ordered pairs are equal only when their corresponding components are equal, the last equation is equivalent to the two numerical equations

$$a + x = c \quad \text{and} \quad b + y = d.$$

Solving these equations for x and y gives $x = c - a$ and $y = d - b$, or $(x, y) = (c - a, d - b)$. ▲

Following Example 1(b), we may observe that the sum $(x, y) + (x, y) + \cdots + (x, y)$, with n terms, is equal to (nx, ny). Thinking of the sum as "n times (x, y)," we denote it by $n(x, y)$, so we have the equation

$$n(x, y) = (nx, ny).$$

Noting that the right-hand side of this equation makes sense when n is any real number, not just a positive integer, we take this as a definition.

Multiplication of Ordered Pairs by Numbers

If (x, y) is an ordered pair and r is a real number, the ordered pair (rx, ry) is called the *product* of r and (x, y) and is denoted by $r(x, y)$. Thus, $r(x, y) = (rx, ry)$.

To distinguish ordinary numbers from ordered pairs, we sometimes call numbers *scalars*. The operation just defined is called *scalar multiplication*. Notice that we have not defined the product of two ordered pairs—we will do so in Section 13.4.

Example 2 (a) Calculate $4(2,-3)+4(3,5)$. (b) Calculate $4[(2,-3)+(3,5)]$.

Solution (a) $4(2,-3)+4(3,5)=(8,-12)+(12,20)=(20,8)$.
(b) $4[(2,-3)+(3,5)]=4(5,2)=(20,8)$. ▲

It is sometimes a useful shorthand to denote an ordered pair by a single letter such as $A=(x,y)$. This makes the algebra of ordered pairs look more like the algebra of ordinary numbers. The results in Example 2 illustrate the following general rule.

Example 3 Show that if a is a number and A_1 and A_2 are ordered pairs, then $a(A_1+A_2)=aA_1+aA_2$.

Solution We write $A_1=(x_1,y_1)$ and $A_2=(x_2,y_2)$. Then $A_1+A_2=(x_1+x_2,y_1+y_2)$, and so

$$\begin{aligned}a(A_1+A_2)&=(a(x_1+x_2),a(y_1+y_2))=(ax_1+ax_2,ay_1+ay_2)\\&=(ax_1,ay_1)+(ax_2,ay_2)=a(x_1,y_1)+a(x_2,y_2)\\&=aA_1+aA_2,\end{aligned}$$

as required. ▲

All the usual algebraic identities which make sense for numbers and ordered pairs are true, and they can be used in computations with ordered pairs (see Exercises 17–22).

Example 4 (a) Find real numbers a_1,a_2,a_3 such that $a_1(3,1)+a_2(6,2)+a_3(-1,1)=(5,6)$.
(b) Is the solution in part (a) unique?
(c) Can you find a solution in which a_1, a_2, and a_3 are integers?

Solution (a) $a_1(3,1)+a_2(6,2)+a_3(-1,1)=(3a_1+6a_2-a_3,a_1+2a_2+a_3)$; for this to equal $(5,6)$, we must solve the equations

$$\begin{aligned}&3a_1+6a_2-a_3=5,\\&a_1+2a_2+a_3=6.\end{aligned}$$

A solution of these equations is $a_1=0$, $a_2=\frac{11}{8}$, $a_3=\frac{13}{4}$. (Part (b) explains where this solution came from.)
(b) We can rewrite the equations as

$$\begin{aligned}&6a_2-a_3=5-3a_1\\&2a_2+a_3=6-a_1.\end{aligned}$$

We may choose a_1 at will, obtaining a pair of linear equations in a_2 and a_3 which always have a solution, since the lines in the (a_2,a_3) plane which they represent have different slopes. The choice $a_1=0$ led us to the equations $6a_2-a_3=5$ and $2a_2+a_3=6$, with the solution as given in part (a). The choice $a_1=6$, for instance, leads to the new solution $a_1=6$, $a_2=-\frac{13}{8}$, $a_3=\frac{13}{4}$, so the solution is not unique.
(c) We notice that the sums $3+1=4$, $6+2=8$, and $-1+1=0$ of the components in each of the ordered pairs are even. Thus the sum $4a_1+8a_2$ of the components of $a_1(3,1)+a_2(6,2)+a_3(-1,1)$ is even if a_1, a_2, and a_3 are integers; but $5+6=11$ is odd, so there is no solution with a_1, a_2, and a_3 being integers. ▲

Example 5 Interpret the chemical equation $2NH_2 + H_2 = 2NH_3$ as a relation in the algebra of ordered pairs.

Solution We think of the molecule N_xH_y (x atoms of nitrogen, y atoms of hydrogen) as represented by the ordered pair (x, y). Then the chemical equation given is equivalent to $2(1,2) + (0,2) = 2(1,3)$. Indeed, both sides are equal to $(2,6)$. ▲

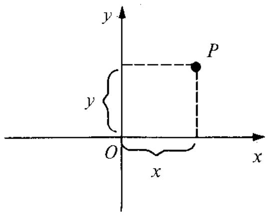


Figure 13.1.1. The point P has coordinates (x, y) relative to the given axes.

We know from Section R.4 how to represent points P in the plane by ordered pairs by selecting an origin O and two perpendicular lines throught it. Relative to these axes, P is assigned the coordinates (x, y), as in Fig. 13.1.1. If the axes are changed, the coordinates of P change as well. (In Section 14.2, we will study how coordinates change when the axes are rotated.) When a definite coordinate system is understood, we refer to "the point (x, y)" when x and y are the coordinates in that system.

We turn now from the algebra of ordered pairs to the related geometric concept of a vector.

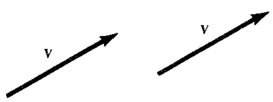


Figure 13.1.2. A vector **v** is an arrow with definite length and direction. The same vector is represented by the two arrows in this figure.

Vectors in the Plane

A *vector* in the plane is a directed line segment in the plane and is drawn as an arrow.

Vectors are denoted by boldface symbols such as **v**. Two directed line segments will be said to be *equal* when they have the same length and direction (as in Fig. 13.1.2).[2]

The vector represented by the arrow from a point P to a point Q is denoted $\overrightarrow{PQ}$. (Figure 13.1.3). If the arrows from P_1 to Q_1 and P_2 to Q_2 represent the same vector, we write $\overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}$.

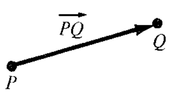


Figure 13.1.3. The vector from P to Q is denoted $\overrightarrow{PQ}$.

Ordered pairs are related to vectors in the following way. We first choose a set of x and y axes. Given a vector **v**, we drop perpendiculars from its head and tail to the x and y axes, as shown in Fig. 13.1.4, producing two signed numbers x and y equaling the directed lengths of the vector in the x and y directions. These numbers are called the *components* of the vector. Notice that once the x and y axes are chosen, the components do not depend on where the arrow representing the vector **v** is placed; they depend only on the magnitude and direction of **v**. Thus, for any vector **v**, we get an ordered pair (x, y). Conversely, given an ordered pair (x, y), we can construct a vector with these components; for example, the vector from the origin to the point with coordinates (x, y). The arrow representing this vector can be relocated as long as its magnitude and direction are preserved.

Operations of vector addition and scalar multiplication are defined in the following box. These geometric definitions will be seen later to be related to the algebraic ones we studied earlier.

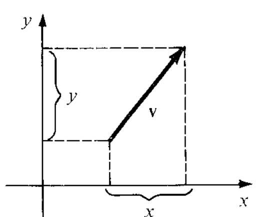


Figure 13.1.4. The components of **v** are x and y.

[2] Strictly speaking, this definition does not make sense. The two directed line segments in Fig. 13.1.2 are clearly *not* equal—that is, they are not identical. However, it is very convenient to have the *set* of all directed line segments with the same magnitude and direction represent a single geometric entity—a *vector*. A convenient way to do this is to *regard* two such segments as equal.

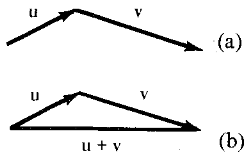


Figure 13.1.5. The geometric construction of $\mathbf{u} + \mathbf{v}$.

Vector Addition and Scalar Multiplication

Addition. Let **u** and **v** be vectors. Their sum is the vector represented by the arrow from the tail of **u** to the tip of **v** when the tail of **v** is placed at the tip of **u** (Fig. 13.1.5).

Scalar Multiplication. Let **u** be a vector and r a number. The vector $r\mathbf{u}$ is an arrow with length $|r|$ times the length of **u**. It has the same direction as **u** if $r > 0$ and the opposite direction if $r < 0$ (Fig. 13.1.6).

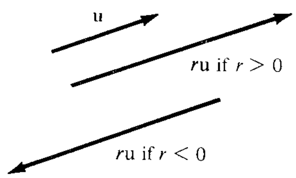


Figure 13.1.6. The product $r\mathbf{u}$.

Example 6 In Fig. 13.1.7, which vector is (a) $\mathbf{u} + \mathbf{v}$?, (b) $3\mathbf{u}$?, (c) $-\mathbf{v}$?

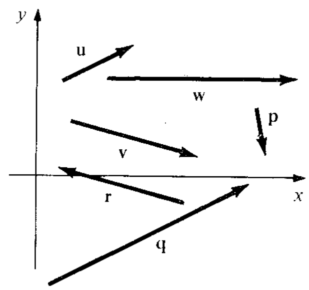


Figure 13.1.7. Find $\mathbf{u} + \mathbf{v}$, $3\mathbf{u}$, and $-\mathbf{v}$.

Solution (a) To construct $\mathbf{u} + \mathbf{v}$, we represent **u** and **v** by directed line segments so that the head of the first coincides with the tail of the second. We fill in the third side of the triangle to obtain $\mathbf{u} + \mathbf{v}$ (see Fig. 13.1.5(b)). Comparing Fig. 13.1.5(b) with Fig. 13.1.7, we find that $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

(b) $3\mathbf{u} = \mathbf{q}$ (see Fig. 13.1.8).

(c) $-\mathbf{v} = (-1)\mathbf{v} = \mathbf{r}$ (see Fig. 13.1.8). ▲

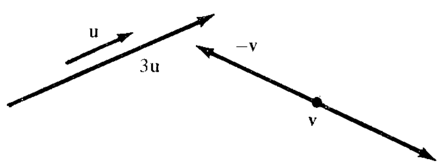


Figure 13.1.8. To find $3\mathbf{u}$, draw a vector in the same direction as **u**, three times as long; $-\mathbf{v}$ is a vector having the same length as **v**, pointing in the opposite direction.

Example 7 Let **u** and **v** be the vectors shown in Fig. 13.1.9. Draw $\mathbf{u} + \mathbf{v}$ and $-2\mathbf{u}$. What are their components?

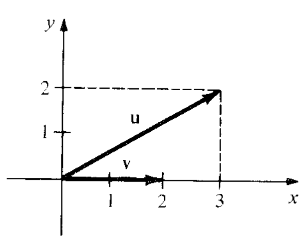


Figure 13.1.9. Find $\mathbf{u} + \mathbf{v}$ and $-2\mathbf{u}$.

Solution We place the tail of **v** at the tip of **u** to obtain the vector shown in Fig. 13.1.10.

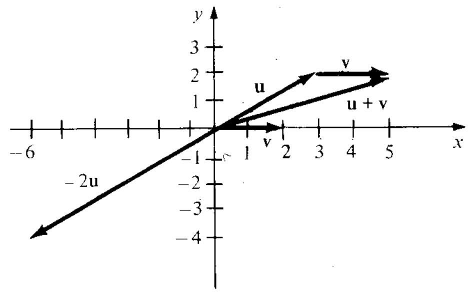


Figure 13.1.10. Computing $\mathbf{u}+\mathbf{v}$ and $-2\mathbf{u}$.

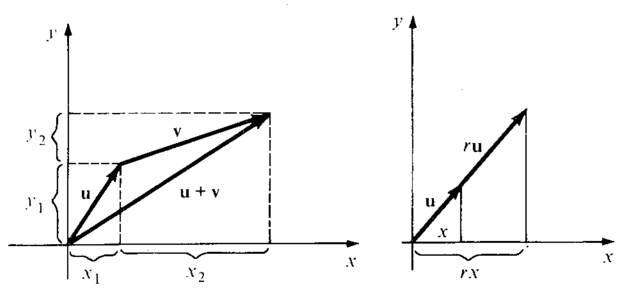
The vector $-2\mathbf{u}$, also shown, has length twice that of **u** and points in the opposite direction. From the figure, we see that $\mathbf{u}+\mathbf{v}$ has components $(5,2)$ and $-2\mathbf{u}$ has components $(-6,-4)$. ▲

The results in Example 7 are illustrations of the following general rules which relate the geometric operations on vectors to the algebra of ordered pairs.

Vectors and Ordered Pairs

Addition. If **u** has components (x_1, y_1) and **v** has components (x_2, y_2), then $\mathbf{u}+\mathbf{v}$ has components (x_1+x_2, y_1+y_2).
Scalar Multiplication. If **u** has components (x, y), then $r\mathbf{u}$ has components (rx, ry).

The statements in this box may be proved by plane geometry. For example, the addition rule follows by an examination of Fig. 13.1.11(a), and the one for scalar multiplication follows from the similarity of the triangles in Fig. 13.1.11(b).



(a) (b)

Figure 13.1.11. The geometry which relates vector algebra to the algebra of ordered pairs.

We can use the correspondence between ordered pairs and vectors to transfer to vectors the identities we know for ordered pairs, such as $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. This identity can also be seen geometrically, as in Fig. 13.1.12, which illustrates another geometric interpretation of vector addition. To add **u** and **v**, we choose representatives $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ having their tails at the same point P. If we complete the figure to a parallelogram $PQSR$, then the diagonal $\overrightarrow{PS}$ represents $\mathbf{u}+\mathbf{v}$. For this reason, physical quantities which combine by vector addition are sometimes said to "obey the parallelogram law."

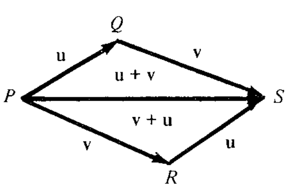


Figure 13.1.12. Illustrating the identity $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ and the parallelogram law of addition.

If **v** and **w** are vectors, their difference $\mathbf{v}-\mathbf{w}$ is the vector such that $(\mathbf{v}-\mathbf{w})+\mathbf{w}=\mathbf{v}$. It follows from the "triangle" construction of vector sums that if we draw **v** and **w** with a common tail, $\mathbf{v}-\mathbf{w}$ is represented by the

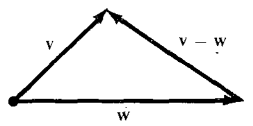


Figure 13.1.13. Geometric interpretation of vector subtraction.

directed line segment from the head of **w** to the head of **v**. (See Fig. 13.1.13.) The components of $\mathbf{v}-\mathbf{w}$ are obtained by subtracting the corresponding ordered pairs. Since $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$ (Fig. 13.1.14), we obtain the following.

Vectors and Directed Line Segments

If the point P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2), then the vector $\overrightarrow{PQ}$ has components $(x_2 - x_1, y_2 - y_1)$.

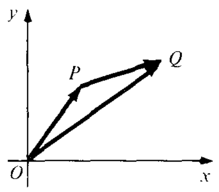


Figure 13.1.14. $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$.

Example 8 (a) Find the components of the vector from $(3,5)$ to $(4,7)$.
(b) Add the vector **v** from $(-1,0)$ to $(2,-3)$ and the vector **w** from $(2,0)$ to $(1,1)$.
(c) Multiply the vector **v** in (b) by 8. If this vector is represented by the directed line segment from $(5,6)$ to Q, what is Q?

Solution (a) By the preceding box, we subtract the ordered pairs $(4,7)-(3,5)=(1,2)$. Thus the required components are $(1,2)$.
(b) The vector **v** has components $(2,-3)-(-1,0)=(3,-3)$ and **w** has components $(1,1)-(2,0)=(-1,1)$. Therefore, the vector $\mathbf{v}+\mathbf{w}$ has components $(3,-3)+(-1,1)=(2,-2)$.
(c) The vector $8\mathbf{v}$ has components $8(3,-3)=(24,-24)$. If this vector is represented by the directed line segment from $(5,6)$ to Q, and Q has coordinates (x,y), then $(x,y)-(5,6)=(24,-24)$, so $(x,y)=(5,6)+(24,-24)=(29,-18)$. ▲

Exercises for Section 13.1

Complete the computations in Exercises 1–4.

1. $(1,2)+(3,7)=$
2. $(-2,6)-6(2,-10)=$
3. $3[(1,1)-2(3,0)]=$
4. $2[(8,6)-4(2,-1)]=$

Solve for the unknown quantities, if possible, in Exercises 5–16.

5. $(1,2)+(0,y)=(1,3)$
6. $r(7,3)=(14,6)$
7. $a(2,-1)=(6,-\pi)$
8. $(7,2)+(x,y)=(3,10)$
9. $2(1,b)+(b,4)=(3,4)$
10. $(x,2)+(-3)(x,y)=(-2x,1)$
11. $0(3,a)=(3,a)$
12. $6(1,0)+b(0,1)=(6,2)$
13. $a(1,1)+b(1,-1)=(3,5)$
14. $(a,1)-(2,b)=(0,0)$
15. $(3a,b)+(b,a)=(1,1)$
16. $a(3a,1)+a(1,-1)=(1,0)$

In Exercises 17–22, A, B, and C denote ordered pairs; O is the pair $(0,0)$; if $A=(x_1,y_1)$, then $-A=(-x_1,-y_1)$; and a and b are numbers. Show the following.

17. $A+O=A$
18. $A+(-A)=O$
19. $(A+B)+C=A+(B+C)$
20. $A+B=B+A$
21. $a(bA)=(ab)A$
22. $(a+b)A=aA+bA$
23. Describe geometrically the set of all points with coordinates of the form $m(0,1)+n(1,1)$, where m and n are integers. (A sketch will do.)
24. Describe geometrically the set of all points with coordinates of the form $m(0,1)+r(1,1)$, where m is an integer and r is a real number.
25. (a) Write the chemical equation $k\mathrm{SO}_3+l\mathrm{S}_2=m\mathrm{SO}_2$ as an equation in ordered pairs.

(b) Write the equation in part (a) as a pair of simultaneous equations in k, l, and m.
(c) Solve the equation in part (b) for the smallest positive integer values of k, l, and m.

26. Illustrate the solution of Exercise 25 by a vector diagram in the plane, with SO_3, S_2, and SO_2 represented as vectors.
27. In Fig. 13.1.15, which vector is (a) $\mathbf{a} - \mathbf{b}$?, (b) $\frac{1}{2}\mathbf{a}$?
28. In Fig. 13.1.15, find the number r such that $\mathbf{c} - \mathbf{a} = r\mathbf{b}$.
29. Trace Fig. 13.1.15 and draw the vectors (a) $\mathbf{c} + \mathbf{d}$, (b) $-2\mathbf{e} + \mathbf{a}$. What are their components?
30. Trace Fig. 13.1.15 and draw the vectors (a) $3(\mathbf{e} - \mathbf{d})$, (b) $-\frac{2}{3}\mathbf{c}$. What are their components?

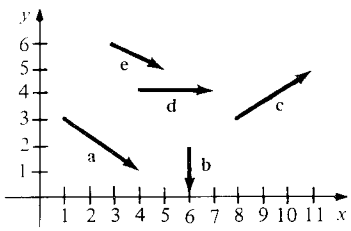


Figure 13.1.15. Compute with these vectors in Exercises 27–30.

In Exercises 31–36, let $\mathbf{u}$ have components $(2, 1)$ and $\mathbf{v}$ have components $(1, 2)$. Draw each of the indicated vectors.

31. $\mathbf{u} + \mathbf{v}$
32. $\mathbf{u} - \mathbf{v}$
33. $2\mathbf{u}$
34. $-4\mathbf{v}$
35. $2\mathbf{u} - 4\mathbf{v}$
36. $-\mathbf{u} + 2\mathbf{v}$
37. Let $P = (2, 1)$, $Q = (3, 3)$, and $R = (4, 1)$ be points in the xy plane.
(a) Draw (on the same diagram) these vectors: $\mathbf{v}$ joining P to Q; $\mathbf{w}$ joining Q to R; $\mathbf{u}$ joining R to P.
(b) What are the components of $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$?
(c) What is $\mathbf{v} + \mathbf{w} + \mathbf{u}$?
38. Answer the questions in Exercise 37 for $P = (-2, -1)$, $Q = (-3, -3)$, and $R = (-1, -4)$.
39. (a) Draw the vector $\mathbf{v}_1$ joining $(1, 0)$ to $(1, 1)$.
(b) What are the components of $\mathbf{v}_1$?
(c) Draw $\mathbf{v}_2$ joining $(1, 0)$ to $(1, \frac{5}{2})$ and find the components of $\mathbf{v}_2$.
(d) Draw the vector $\mathbf{v}_3$ joining $(1, 0)$ to $(1, -2)$.
(e) What are the coordinates of an arbitrary point on the vertical (that is, parallel to the y axis) line through $(1, 0)$?
(f) What are the components of the vector $\mathbf{v}$ joining $(1, 0)$ to such a point?
40. (a) Draw a vector $\mathbf{v}$ joining $(-1, 1)$ to $(1, 1)$.
(b) What are the components of $\mathbf{v}$?
(c) Sketch the vectors $\mathbf{v}_t = (-1, 1) + t\mathbf{v}$ when $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1.
(d) Describe, geometrically, the set of vectors $\mathbf{v}_t = (-1, 1) + t\mathbf{v}$, where t takes on all values between 0 and 1. (Assume that all the vectors have their tails at the origin.)

★41. We say that $\mathbf{v}$ and $\mathbf{w}$ are *linearly dependent* if there are numbers r and s, not both zero, such that $r\mathbf{v} + s\mathbf{w} = \mathbf{0}$. Otherwise $\mathbf{v}$ and $\mathbf{w}$ are called *linearly independent*.
(a) Are $(0, 0)$ and $(1, 1)$ linearly dependent?
(b) Show that two non-zero vectors are linearly dependent if and only if they are parallel.
(c) Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in the plane given by $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$. Show that $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent if and only if $ad = bc$. [*Hint*: For one implication, you might use three cases: $b \neq 0$, $d \neq 0$, and $b = d = 0$.]
(d) Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors in the plane which are linearly independent. Show that for any vector $\mathbf{u}$ in the plane there are numbers x and y such that $x\mathbf{v} + y\mathbf{w} = \mathbf{u}$.

★42. Let $P = (a, b)$ and $Q = (c, d)$ be points in the plane. (You may assume that $0 < a < c$ and $b > d > 0$ to make the picture unambiguous.) Compute the area of the parallelogram with vertices at O, P, Q, and $P + Q$. Comment on the relationship between this and Exercise 41(d).

13.2 Vectors in Space

A vector in space has three components.

The plane is two-dimensional, but space is three-dimensional—that is, it requires *three* numbers to specify the position of the point in space. For instance, the location of a bird is specified not only by the two coordinates of the point on the ground directly below it, but also by its height. Accordingly, our algebraic model for space will be the set of *triples* (x, y, z) rather than pairs of real numbers.

If one starts with abstract "space" as studied in elementary solid geome-

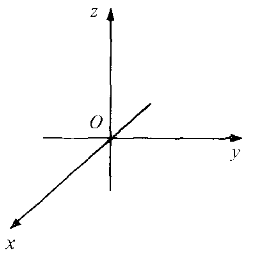


Figure 13.2.1. Coordinate axes in space.

try, the first step in the introduction of coordinates is the choice of an origin O and three directed lines, each perpendicular to the other two, called the x, y, and z axes. We will usually draw figures in space with the axes oriented as in Fig. 13.2.1.

Think of the x axis as pointing toward you, out of the paper. Notice that if you wrap the fingers of your right hand around the z axis, with your fingers curling in the usual (counterclockwise) direction of rotation in the xy plane, then your thumb points toward the positive z axis. For this reason, we say that the choice of axes obeys the *right-hand rule*. For example, the coordinate axes (a) and (d) in Fig. 13.2.2 obey the right-hand rule, but (b) and (c) do not. (Think of all horizontal and vertical arrows as being in the plane of the paper, while slanted arrows point out toward you.)

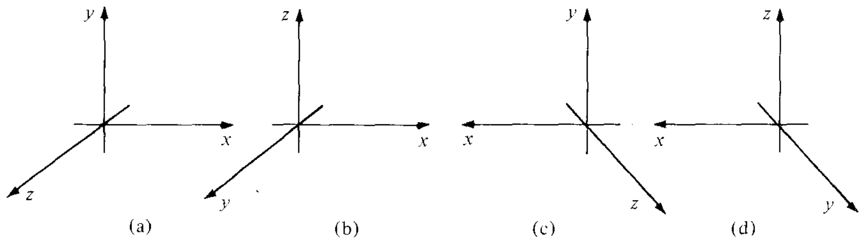


Figure 13.2.2. Which axes obey the right-hand rule?

Given a point P in space, drop a perpendicular from P to each of the axes. By measuring the (directed) distance from the origin to the foot of each of these perpendiculars, we obtain numbers (x, y, z) which we call the *coordinates* of P (see Fig. 13.2.3).

If you cannot see the lines through P in Fig. 13.2.3 as being perpendicular to the axis it may help to draw in some additional lines *parallel* to the axes, using the convention that lines which are parallel to one another in space are drawn parallel. This simple convention, sometimes called the *rule of parallel projection*, does not conform to ordinary rules of perspective (think of railroad tracks "converging at infinity"), but it is reasonably accurate if your distance from an object is great compared to the size of the object.

Now look at Fig. 13.2.4. Observe that the point Q, obtained by dropping a perpendicular from P to the xy plane, has coordinates $(x, y, 0)$. Similarly, the points R and S, obtained by dropping perpendiculars from P to the yz and xz planes, have coordinates $(0, y, z)$ and $(x, 0, z)$, respectively. The coordinates of T, U, and V are $(x, 0, 0)$, $(0, y, 0)$, and $(0, 0, z)$.

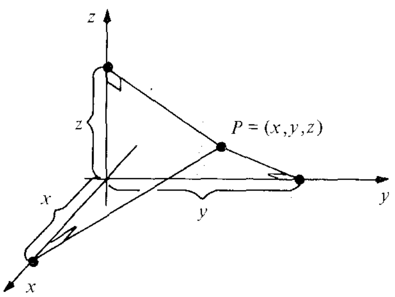


Figure 13.2.3. We obtain the coordinates of the point P by dropping perpendiculars to the x, y, and z axes.

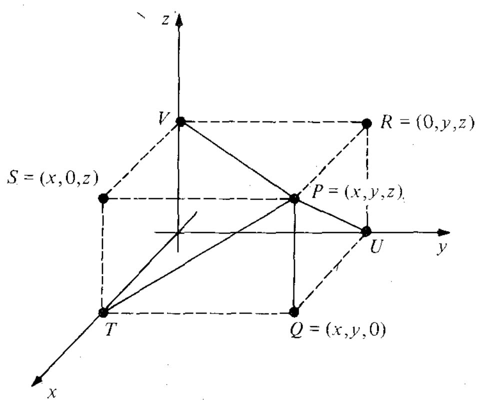


Figure 13.2.4. Lines which are parallel in space are drawn parallel.

As in the plane, we can find the unique point which has a given ordered triple (x, y, z) as its coordinates. To do so, we begin by finding the point $Q = (x, y, 0)$ in the xy plane. By drawing a line through Q and parallel to the z axis, we can locate the point P at a (directed) distance of z units from Q along this line. The process just described can be carried out graphically, as in the following example.

Example 1 Plot the point $(1,2,-3)$.

Solution We begin by plotting $(1,2,0)$ in the xy plane (Fig. 13.2.5(a)). Then we draw the line through this point parallel to the z axis and measure 3 units downward (Fig. 13.2.5(b)). ▲

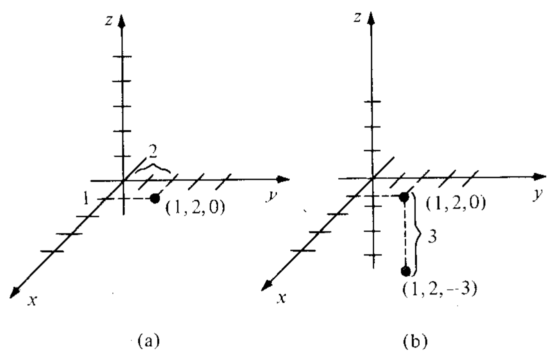


Figure 13.2.5. Plotting the point $(1,2,-3)$.

Warning If you are given a picture consisting simply of three axes (with units of measure) and a point, it is not possible to determine the coordinates of the point from these data alone, since some information must be lost in making a two-dimensional picture of the three-dimensional space. (See Review Exercise 86.)

Addition and scalar multiplication are defined for ordered triples just as for pairs.

The Algebra of Ordered Triples

1. If (x_1, y_1, z_1) and (x_2, y_2, z_2) are ordered triples of real numbers, the ordered triple $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is called their *sum* and is denoted by $(x_1, y_1, z_1) + (x_2, y_2, z_2)$.
2. If (x, y, z) is an ordered triple and r is a real number, the triple (rx, ry, rz) is called the *product* of r and (x, y, z) and is denoted by $r(x, y, z)$.

Example 2 Find $(3,2,-2)+(-1,-2,-1)$ and $(-6)(2,-1,1)$.

Solution We have $(3,2,-2)+(-1,-2,-1)=(3-1,2-2,-2-1)=(2,0,-3)$ and $(-6)(2,-1,1)=(-12,6,-6)$. ▲

Now we look at vectors in space, the geometric objects which correspond to ordered triples.

Vectors in Space

A *vector* in space is a directed line segment in space and is drawn as an arrow. Two directed line segments will be regarded as equal when they have the same length and direction.

Vectors are denoted by boldface symbols. The vector represented by the arrow from a point P to a point Q is denoted $\overrightarrow{PQ}$. If the arrows from P_1 to Q_1 and P_2 to Q_2 represent the same vectors, we write $\overrightarrow{P_1Q_1} = \overrightarrow{P_2Q_2}$.

Vectors in space are related to ordered triples as follows. We choose x, y, and z axes and drop perpendiculars to the three axes. The directed distances obtained are called the *components* of the vector (see Fig. 13.2.6).

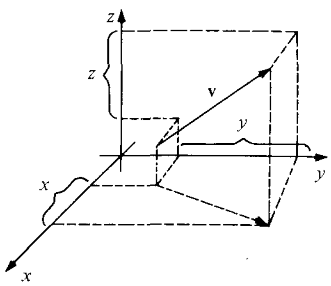


Figure 13.2.6. The vector **v** has components (x, y, z).

Vector addition and scalar multiplication for vectors in space are defined just as in the plane. The student should reread the corresponding development in Section 13.1, replacing the plane by space.

Vectors and Ordered Triples

1. The algebra of vectors corresponds to the algebra of ordered triples.
2. If P has coordinates (x_1, y_1, z_1) and Q has coordinates (x_2, y_2, z_2), then the vector $\overrightarrow{PQ}$ has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Example 3 (a) Sketch $-2\mathbf{v}$, where $\mathbf{v}$ has components $(-1, 1, 2)$. (b) If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, show that $\mathbf{v} - \frac{1}{3}\mathbf{w}$ and $3\mathbf{v} - \mathbf{w}$ are parallel.

Solution (a) The vector $-2\mathbf{v}$ is twice as long as $\mathbf{v}$ but points in the opposite direction (see Fig. 13.2.7). (b) $\mathbf{v} - \frac{1}{3}\mathbf{w} = \frac{1}{3}(3\mathbf{v} - \mathbf{w})$; vectors which are multiples of one another are parallel. ▲

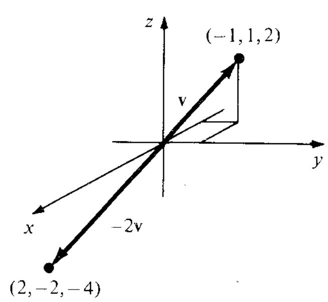


Figure 13.2.7. Multiplying $(-1, 1, 2)$ by -2.

Example 4 Let $\mathbf{v}$ be the vector with components $(3, 2, -2)$ and let $\mathbf{w}$ be the vector from the point $(2, 1, 3)$ to the point $(-1, 0, -1)$. Find $\mathbf{v} + \mathbf{w}$. Illustrate with a sketch.

Solution Since $\mathbf{w}$ has components $(-1, 0, -1) - (2, 1, 3) = (-3, -1, -4)$, we find that $\mathbf{v} + \mathbf{w}$ has components $(3, 2, -2) + (-3, -1, -4) = (0, 1, -6)$, as illustrated in Fig. 13.2.8. ▲

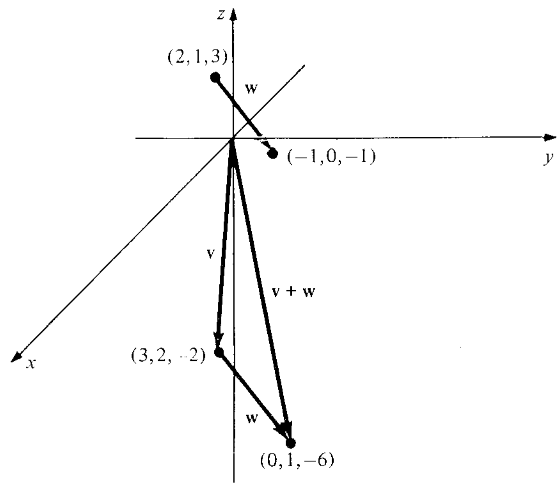


Figure 13.2.8. Adding $\mathbf{v} = (3, 2, -2)$ to $\mathbf{w}$, the vector from $(2, 1, 3)$ to $(-1, 0, -1)$.

To describe vectors in space, it is convenient to introduce three special vectors along the x, y, and z axes.

$\mathbf{i}$: the vector with components $(1, 0, 0)$;
$\mathbf{j}$: the vector with components $(0, 1, 0)$;
$\mathbf{k}$: the vector with components $(0, 0, 1)$.

These *standard basis vectors* are illustrated in Fig. 13.2.9. In the plane one has, analogously, $\mathbf{i}$ and $\mathbf{j}$ with components $(1, 0)$ and $(0, 1)$.

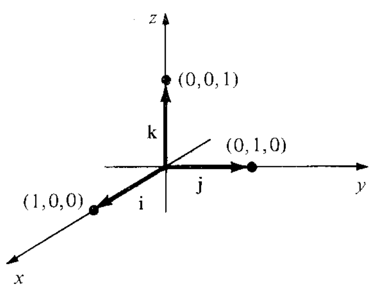


Figure 13.2.9. The standard basis vectors.

Now, let $\mathbf{v}$ be any vector, and let (a, b, c) be its components. Then

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k},$$

since the right-hand side is given in components by

$$a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = (a, 0, 0) + (0, b, 0) + (0, 0, c) = (a, b, c).$$

Thus we can express every vector as a sum of scalar multiples of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The Standard Basis Vectors

1. The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the three coordinate axes, as shown in Fig. 13.2.9.
2. If $\mathbf{v}$ has components (a, b, c), then

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Example 5 (a) Express the vector whose components are $(e, \pi, -\sqrt{3})$ in the standard basis. (b) Express the vector $\mathbf{v}$ joining $(2,0,1)$ to $(\frac{3}{2}, \pi, -1)$ by using the standard basis.

Solution (a) $\mathbf{v} = e\mathbf{i} + \pi\mathbf{j} - \sqrt{3}\,\mathbf{k}$. (b) The vector $\mathbf{v}$ has components $(\frac{3}{2}, \pi, -1) - (2,0,1) = (-\frac{1}{2}, \pi, -2)$, so $\mathbf{v} = -\frac{1}{2}\mathbf{i} + \pi\mathbf{j} - 2\mathbf{k}$. ▲

Now we turn to some physical applications of vectors.[3] A simple example of a physical quantity represented by a vector is a displacement. Suppose that, on a part of the earth's surface small enough to be considered flat, we introduce coordinates so that the x axis points east, the y axis points north, and the unit of length is the kilometer. If we are at a point P and wish to get to a point Q, the *displacement vector* $\mathbf{u}$ joining P to Q tells us the direction and distance we have to travel. If x and y are the components of this vector, the displacement of Q from P is "x kilometers east, y kilometers north."

Example 6 Suppose that two navigators, who cannot see one another but can communicate by radio, wish to determine the relative position of their ships. Explain how they can do this if they can each determine their displacement vector to the same lighthouse.

Solution Let P_1 and P_2 be the positions of the ships and Q be the position of the lighthouse. The displacement of the lighthouse from the ith ship is the vector $\mathbf{u}_i$ joining P_i to Q. The displacement of the second ship from the first is the vector $\mathbf{v}$ joining P_1 to P_2. We have $\mathbf{v} + \mathbf{u}_2 = \mathbf{u}_1$ (Fig. 13.2.10), so $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$.

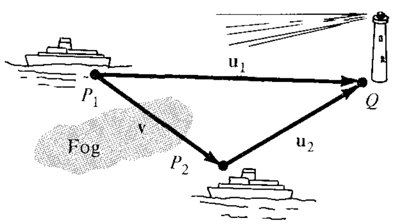


Figure 13.2.10. Vector methods can be used to locate objects.

That is, the displacement from one ship to the other is the difference of the displacements from the ships to the lighthouse. ▲

We can also represent the velocity of a moving object as a vector. For the moment, we will consider only objects moving at uniform speed along straight lines—the general case is discussed in Section 14.6. Suppose, for example, that a boat is steaming across a lake at 10 kilometers per hour in the northeast direction. After 1 hour of travel, the displacement is $(10/\sqrt{2}, 10/\sqrt{2}) \approx (7.07, 7.07)$ (see Fig. 13.2.11). The vector whose components are $(10/\sqrt{2}, 10/\sqrt{2})$ is called the *velocity vector* of the boat. In general, if an object is moving uniformly along a straight line, *its velocity vector is the displacement vector from the position at any moment to the position 1 unit of time later*. If a

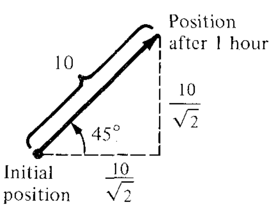


Figure 13.2.11. If an object moves northeast at 10 kilometers per hour, its velocity vector has components $(10/\sqrt{2}, 10/\sqrt{2})$.

[3] Historical note: Many scientists resisted the use of vectors in favor of the more complicated theory of quaternions until around 1900. The book which popularized vector methods was *Vector Analysis*, by E. B. Wilson (reprinted by Dover in 1960), based on lectures of J. W. Gibbs at Yale in 1899–1900. Wilson was reluctant to take Gibbs' course since he had just completed a full-year course in quaternions at Harvard under J. M. Pierce, a champion of quaternionic methods, but was forced by a dean to add the course to his program. (For more details, see M. J. Crowe, *A History of Vector Analysis*, University of Notre Dame Press, 1967.)

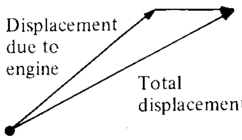


Figure 13.2.12. The total displacement is the sum of the displacements due to the engine and the current.

current appears on the lake, moving due eastward at 2 kilometers per hour, and the boat continues to point in the same direction with its engine running at the same rate, its displacement after 1 hour will have components given by $(10/\sqrt{2}+2, 10/\sqrt{2})$. (See Fig. 13.2.12.) The new velocity vector, therefore, has components $(10/\sqrt{2}+2, 10/\sqrt{2})$. We note that this is the sum of the original velocity vector $(10/\sqrt{2}, 10/\sqrt{2})$ of the boat and the velocity vector $(2,0)$ of the current.

Similarly, consider a seagull which flies in calm air with velocity vector $\mathbf{v}$. If a wind comes up with velocity $\mathbf{w}$ and the seagull continues flying "the same way," its actual velocity will be $\mathbf{v}+\mathbf{w}$. One can "see" the direction of the vector $\mathbf{v}$ because it points along the "axis" of the seagull; by comparing the direction of actual motion with the direction of $\mathbf{v}$, you can get an idea of the wind direction (see Fig. 13.2.13).

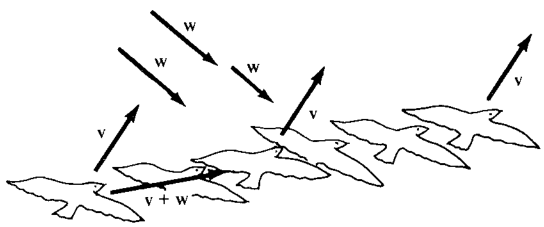


Figure 13.2.13. The velocity $\mathbf{w}$ of the wind can be estimated by comparing the "wingflap" velocity $\mathbf{v}$ with the actual velocity $\mathbf{v}+\mathbf{w}$.

Another example comes from medicine. An electrocardiograph detects the flow of electricity in the heart. Both the magnitude and the direction of the net flow are of importance. This information can be summarized at every instant by means of a vector called the *cardiac vector*. The motion of this vector (see Fig. 13.2.14) gives physicians useful information about the heart's function.[4]

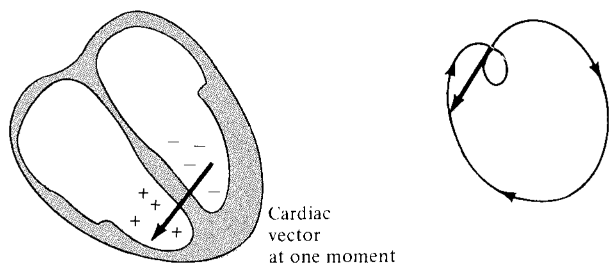


Figure 13.2.14. The magnitude and direction of electrical flow in the heart are indicated by the cardiac vector.

Example 7 A bird is flying in a straight line with velocity vector $10\mathbf{i}+6\mathbf{j}+\mathbf{k}$ (in kilometers per hour). Suppose that (x, y) are coordinates on the ground and z is the height above the ground.

(a) If the bird is at position $(1,2,3)$ at a certain moment, where is it 1 minute later?
(b) How many seconds does it take the bird to climb 10 meters?

Solution (a) The displacement vector from $(1,2,3)$ is $\frac{1}{60}(10\mathbf{i}, 6\mathbf{j}, \mathbf{k}) = \frac{1}{6}\mathbf{i}+\frac{1}{10}\mathbf{j}+\frac{1}{60}\mathbf{k}$, so the new position is $(1,2,3)+(\frac{1}{6},\frac{1}{10},\frac{1}{60}) = (1\frac{1}{6}, 2\frac{1}{10}, 3\frac{1}{60})$.
(b) After t seconds ($= t/3600$ hours), the displacement vector from $(1,2,3)$ is $(t/3600)(10\mathbf{i}+6\mathbf{j}+\mathbf{k}) = (t/360)\mathbf{i}+(t/600)\mathbf{j}+(t/3600)\mathbf{k}$. The increase in altitude is the z component $t/3600$. This will equal 10 meters ($=\frac{1}{100}$ kilometer) when $t/3600 = 1/100$—that is, when $t = 36$ seconds. ▲

[4] See M. J. Goldman, *Principles of Clinical Electrocardiography*, 8th edition, Lange, 1973, Chapters 14 and 19.

Example 8 Physical forces have magnitude and direction and may thus be represented by vectors. If several forces act at once on an object, the resultant force is represented by the sum of the individual force vectors. Suppose that forces $\mathbf{i}+\mathbf{k}$ and $\mathbf{j}+\mathbf{k}$ are acting on a body. What third force must we impose to counteract these two—that is, to make the total force equal to zero?

Solution The force $\mathbf{v}$ should be chosen so that $(\mathbf{i}+\mathbf{k})+(\mathbf{j}+\mathbf{k})+\mathbf{v}=\mathbf{0}$; that is $\mathbf{v}=-(\mathbf{i}+\mathbf{k})-(\mathbf{j}+\mathbf{k})=-\mathbf{i}-\mathbf{j}-2\mathbf{k}$. (Here $\mathbf{0}$ is the *zero vector*, the vector whose components are all zero.) ▲

Exercises for Section 13.2

Plot the points in Exercises 1–4.

1. $(1,0,0)$
2. $(0,2,4)$
3. $(3,-1,5)$
4. $(2,-1,\frac{1}{2})$

Complete the computations in Exercises 5–8.

5. $(6,0,5)+(5,0,6)=$
6. $(0,0,0)+(0,0,0)=$
7. $(1,3,5)+4(-1,-3,-5)=$
8. $(2,0,1)-8(3,-\frac{1}{2},\frac{1}{4})=$

9. Sketch $\mathbf{v}$, $2\mathbf{v}$, and $-\mathbf{v}$, where $\mathbf{v}$ has components $(1,-1,-1)$.
10. Sketch $\mathbf{v}$, $3\mathbf{v}$, and $-\frac{1}{2}\mathbf{v}$, where $\mathbf{v}$ has components $(2,-1,1)$.
11. Let $\mathbf{v}$ have components $(0,1,1)$ and $\mathbf{w}$ have components $(1,1,0)$. Find $\mathbf{v}+\mathbf{w}$ and sketch.
12. Let $\mathbf{v}$ have components $(2,-1,1)$ and $\mathbf{w}$ have components $(1,-1,-1)$. Find $\mathbf{v}+\mathbf{w}$ and sketch.

In Exercises 13–20, express the given vector in terms of the standard basis.

13. The vector with components $(-1,2,3)$.
14. The vector with components $(0,2,2)$.
15. The vector with components $(7,2,3)$.
16. The vector with components $(-1,2,\pi)$.
17. The vector from $(0,1,2)$ to $(1,1,1)$.
18. The vector from $(3,0,5)$ to $(2,7,6)$.
19. The vector from $(1,0,0)$ to $(2,-1,1)$.
20. The vector from $(1,0,0)$ to $(3,-2,2)$.

21. A ship at position $(1,0)$ on a nautical chart (with north in the positive y direction) sights a rock at position $(2,4)$. What is the vector joining the ship to the rock? What angle does this vector make with due north? This is called the *bearing* of the rock from the ship.
22. Suppose that the ship in Exercise 21 is pointing due north and travelling at a speed of 4 knots relative to the water. There is a current flowing due east at 1 knot. (The units on the chart are nautical miles; 1 knot = 1 nautical mile per hour.)
    (a) If there were no current, what vector $\mathbf{u}$ would represent the velocity of the ship relative to the sea bottom?
    (b) If the ship were just drifting with the current, what vector $\mathbf{v}$ would represent its velocity relative to the sea bottom?
    (c) What vector $\mathbf{w}$ represents the total velocity of the ship?
    (d) Where would the ship be after 1 hour?
    (e) Should the captain change course?
    (f) What if the rock were an iceberg?
23. An airplane is located at position $(3,4,5)$ at noon and travelling with velocity $400\mathbf{i}+500\mathbf{j}-\mathbf{k}$ kilometers per hour. The pilot spots an airport at position $(23,29,0)$.
    (a) At what time will the plane pass directly over the airport? (Assume that the earth is flat and that the vector $\mathbf{k}$ points straight up.)
    (b) How high above the airport will the plane be when it passes?
24. The wind velocity $\mathbf{v}_1$ is 40 miles per hour from east to west while an airplane travels with air speed $\mathbf{v}_2$ of 100 miles per hour due north. The speed of the airplane relative to the earth is the vector sum $\mathbf{v}_1+\mathbf{v}_2$.
    (a) Find $\mathbf{v}_1+\mathbf{v}_2$.
    (b) Draw a figure to scale.
25. A force of 50 lbs is directed 50° above horizontal, pointing to the right. Determine its horizontal and vertical components. Display all results in a figure.
26. Two persons pull horizontally on ropes attached to a post, the angle between the ropes being 60°. A pulls with a force of 150 lbs, while B pulls with a force of 110 lbs.
    (a) The resultant force is the vector sum of the two forces in a conveniently chosen coordinate system. Draw a figure to scale which graphically represents the three forces.
    (b) Using trigonometry, determine formulas for the vector components of the two forces in a conveniently chosen coordinate system. Perform the algebraic addition, and find the angle the resultant force makes with A.
27. What restrictions must be placed on x, y, and z so that the triple (x,y,z) will represent a point

on the y axis? On the z axis? In the xy plane? In the xz plane?

28. Plot on one set of axes the eight points of the form (a, b, c), where a, b, and c are each equal to 1 or -1. Of what geometric figure are these the vertices?
29. Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Sketch the vectors $\mathbf{u}$, $2\mathbf{u}$, and $-3\mathbf{u}$ on the same set of axes.

In Exercises 30–34, consider the vectors $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Express the given vector in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

30. $\mathbf{v} + \mathbf{w}$
31. $3\mathbf{v}$
32. $-2\mathbf{w}$
33. $6\mathbf{v} + 8\mathbf{w}$
34. the vector $\mathbf{u}$ from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$. (Assume that the tails of $\mathbf{w}$ and $\mathbf{v}$ are at the same point.)

In Exercises 35–37, let $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j}$. Find numbers a and b such that $a\mathbf{v} + b\mathbf{w}$ is the given vector.

35. $\mathbf{i}$
36. $\mathbf{j}$
37. $3\mathbf{i} + 7\mathbf{j}$
38. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = \mathbf{i}$. Given numbers r, s, and t, find a, b, and c such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$.
39. A 1-kilogram mass located at the origin is suspended by ropes attached to the points $(1, 1, 1)$ and $(-1, -1, 1)$. If the force of gravity is pointing in the direction of the vector $-\mathbf{k}$, what is the vector describing the force along each rope? [*Hint*: Use the symmetry of the problem. A 1-kilogram mass weighs 9.8 newtons.]
40. Write the chemical equation $CO + H_2O = H_2 + CO_2$ as an equation in ordered triples, and illustrate it by a vector diagram in space.
41. (a) Write the chemical equation $pC_3H_4O_3 + qO_2 = rCO_2 + sH_2O$ as an equation in ordered triples with unknown coefficients p, q, r, and s.
    (b) Find the smallest integer solution for p, q, r, and s.
    (c) Illustrate the solution by a vector diagram in space.
42. Suppose that the cardiac vector is given by $\cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$ at time t.
    (a) Draw the cardiac vector for $t = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4, 2\pi$.
    (b) Describe the motion of the tip of the cardiac vector in space if the tail is fixed at the origin.

★43. Let $P_t = (1, 0, 0) + t(2, 1, 1)$, where t is a real number.
    (a) Compute the coordinates of P_t for $t = -1$, 0, 1, and 2.
    (b) Sketch these four points on the same set of axes.
    (c) Try to describe geometrically the set of all the P_t.

★44. The z coordinate of the point P in Fig. 13.2.15 is 3. What are the x and y coordinates?

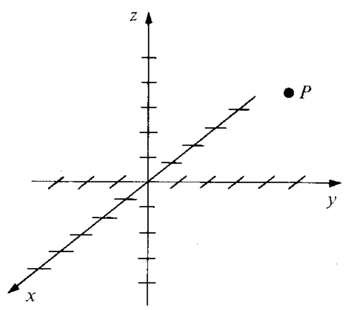


Figure 13.2.15. Let $P = (x, y, 3)$. What are x and y?

13.3 Lines and Distance

Algebraic operations on vectors can be used to solve geometric problems.

In this section we apply the algebra of vectors to the description of lines and planes in space and to the solution of other geometric problems.

The invention of analytic geometry made it possible to solve geometric problems in the plane or space by reducing them to algebraic problems involving number pairs or triples. Vector methods also convert geometric problems to algebraic ones; moreover, the vector calculations are often simpler than those from analytic geometry, since we do not need to write down all the components.

Example 1 Use vector methods to prove that the diagonals of a parallelogram bisect each other.

Solution Let $PQRS$ be the parallelogram, $\mathbf{w}$ the vector $\overrightarrow{PQ}$ and $\mathbf{v}$ the vector $\overrightarrow{PS}$ (see Fig.

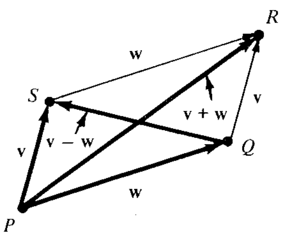


Figure 13.3.1. The diagonals of a parallelogram bisect each other.

13.3.1). Since $PQRS$ is a parallelogram, $\mathbf{v} + \mathbf{w}$ is the vector $\overrightarrow{PR}$.

The vector joining P to the midpoint M_1 of the diagonal PR is thus $\frac{1}{2}(\mathbf{v} + \mathbf{w})$. On the other hand, the vector $\overrightarrow{QS}$ is $\mathbf{v} - \mathbf{w}$, so the vector joining Q to the midpoint M_2 of the diagonal QS is $\frac{1}{2}(\mathbf{v} - \mathbf{w})$.

To show that the diagonals bisect each other, it is enough to show that the midpoints M_1 and M_2 are the same. The vector $\overrightarrow{PM_2}$ is the sum

$$\mathbf{w} + \tfrac{1}{2}(\mathbf{v} - \mathbf{w}) = \mathbf{w} + \tfrac{1}{2}\mathbf{v} - \tfrac{1}{2}\mathbf{w} = \mathbf{w} - \tfrac{1}{2}\mathbf{w} + \tfrac{1}{2}\mathbf{v}$$
$$= \tfrac{1}{2}\mathbf{w} + \tfrac{1}{2}\mathbf{v} = \tfrac{1}{2}(\mathbf{v} + \mathbf{w})$$

which is the same as the vector $\overrightarrow{PM_1}$. It follows that M_1 and M_2 are the same point. ▲

Example 2 Consider the cube in space with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, and $(0,1,1)$. Use vector methods to locate the point one-third of the way from the origin to the middle of the face whose vertices are $(0,1,0)$, $(0,1,1)$, $(1,1,1)$, and $(1,1,0)$.

Solution Refer to Fig. 13.3.2. The vector $\overrightarrow{OP}$ is $\mathbf{j}$, and vector $\overrightarrow{OQ}$ is $\mathbf{i} + \mathbf{j} + \mathbf{k}$. The vector

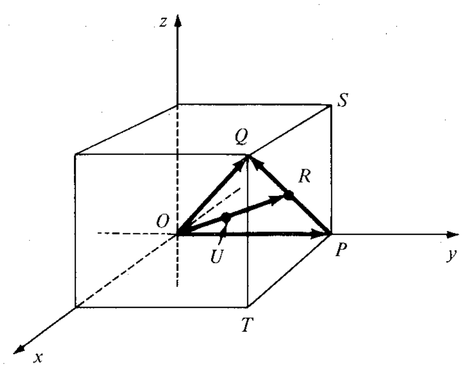


Figure 13.3.2. The point U is one-third of the way from O to the midpoint of the face $PSQT$.

$\overrightarrow{PQ}$ is the difference $\mathbf{i} + \mathbf{j} + \mathbf{k} - \mathbf{j} = \mathbf{i} + \mathbf{k}$; the vector joining P to the midpoint R of PQ (and hence of the face $PSQT$) is one-half of this, that is, $\frac{1}{2}(\mathbf{i} + \mathbf{k})$; the vector $\overrightarrow{OR}$ is then $\mathbf{j} + \frac{1}{2}(\mathbf{i} + \mathbf{k})$, and the vector joining O to the point U one-third of the way from O to R is $\frac{1}{3}[\mathbf{j} + \frac{1}{2}(\mathbf{i} + \mathbf{k})] = \frac{1}{3}\mathbf{j} + \frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{k} = \frac{1}{6}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{6}\mathbf{k}$. It follows that the coordinates of U are $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$. ▲

Example 3 Prove that the figure obtained by joining the midpoints of successive sides of any quadrilateral is a parallelogram.

Solution Refer to Fig. 13.3.3. Let $PQRS$ be the quadrilateral, $\mathbf{v} = \overrightarrow{PS}$, $\mathbf{w} = \overrightarrow{SR}$, $\mathbf{t} = \overrightarrow{PQ}$,

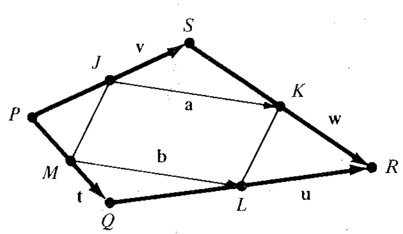


Figure 13.3.3. The figure obtained by joining the midpoints of successive sides of $PQRS$ is the parallelogram $JKLM$.

and $\mathbf{u} = \overrightarrow{QR}$. The vector $\mathbf{a}$ from the midpoint J of PS to the midpoint K of SR satisfies $\frac{1}{2}\mathbf{v} + \mathbf{a} = \mathbf{v} + \frac{1}{2}\mathbf{w}$; solving for $\mathbf{a}$ gives $\mathbf{a} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w} = \frac{1}{2}(\mathbf{v} + \mathbf{w})$. Similarly, the vector $\mathbf{b}$ from the midpoint M of PQ to the midpoint L of QR satisfies $\frac{1}{2}\mathbf{t} + \mathbf{b} = \mathbf{t} + \frac{1}{2}\mathbf{u}$, so $\mathbf{b} = \frac{1}{2}\mathbf{t} + \frac{1}{2}\mathbf{u} = \frac{1}{2}(\mathbf{t} + \mathbf{u})$.

To show that $JKLM$ is a parallelogram, it suffices to show that the vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, but $\mathbf{v}+\mathbf{w}=\mathbf{t}+\mathbf{u}$, since both sides are equal to the vector from P to R, so $\mathbf{a}=\frac{1}{2}(\mathbf{v}+\mathbf{w})=\frac{1}{2}(\mathbf{t}+\mathbf{u})=\mathbf{b}$. ▲

In Section R.4 we discussed the equations of lines in the plane. These equations can be conveniently described in terms of vectors, and this description is equally applicable whether the line is the plane or in space. We will now find such equations in parametric form (see Section 10.4 for a discussion of parametric curves).

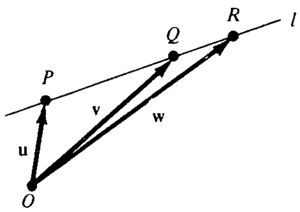


Figure 13.3.4. Since P, Q, and R lie on a line, the vector $\mathbf{w}-\mathbf{u}$ is a multiple of $\mathbf{v}-\mathbf{u}$.

Suppose that we wish to find the equation of the line l passing through the two points P and Q. Let O be the origin and let $\mathbf{u}$ and $\mathbf{v}$ be the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as in Fig. 13.3.4. Let R be an arbitrary point on l and let $\mathbf{w}$ be the vector $\overrightarrow{OR}$. Since R is on l, the vector $\mathbf{w}-\mathbf{u}=\overrightarrow{PR}$ is a multiple of the vector $\mathbf{v}-\mathbf{u}=\overrightarrow{PQ}$—that is, $\mathbf{w}-\mathbf{u}=t(\mathbf{v}-\mathbf{u})$ for some number t. This gives $\mathbf{w}=\mathbf{u}+t(\mathbf{v}-\mathbf{u})=(1-t)\mathbf{u}+t\mathbf{v}$.

The coordinates of the points P, Q, and R are the same as the components of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, so we obtain the parametric equation $R=(1-t)P+tQ$ for the line l.

Parametric Equation of a Line: Point-Point Form

The equation of the line l through the points $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ is

$$R=(1-t)P+tQ.$$

In coordinate form, one has the three equations

$$x=(1-t)x_1+tx_2,$$
$$y=(1-t)y_1+ty_2,$$
$$z=(1-t)z_1+tz_2,$$

where $R=(x,y,z)$ is the typical point of l, and the parameter t takes on all real values.

Example 4 Find the equation of the line through $(2,1,-3)$ and $(6,-1,-5)$.

Solution We have $P=(2,1,-3)$ and $Q=(6,-1,-5)$, so

$$\begin{aligned}(x,y,z)=R=(1-t)P+tQ&=(1-t)(2,1,-3)+t(6,-1,-5)\\&=(2-2t,1-t,-3+3t)+(6t,-t,-5t)\\&=(2+4t,1-2t,-3-2t)\end{aligned}$$

or, since corresponding entries of equal ordered triples are equal,

$$x=2+4t,\qquad y=1-2t,\qquad z=-3-2t. \ ▲$$

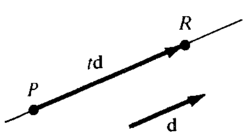


Figure 13.3.5. If the line through P and R has the direction of the vector $\mathbf{d}$, then the vector from P to R is a multiple of $\mathbf{d}$.

We can also ask for the equation of the line which passes through a given point P in the direction of a given vector $\mathbf{d}$. A point R lies on the line (see Fig. 13.3.5) if and only if the vector $\overrightarrow{PR}$ is a multiple of $\mathbf{d}$. Thus we can describe all points R on the line by $\overrightarrow{PR}=t\mathbf{d}$ for some number t. As t varies, R moves on the line; when $t=0$, R coincides with P. Since $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}$, we can rewrite the equation as $\overrightarrow{OR}=\overrightarrow{OP}+t\mathbf{d}$. This reasoning leads to the following conclusion.

Parametric Equation of a Line: Point-Direction Form

The equation of the line through the point $P = (x_0, y_0, z_0)$ and pointing in the direction of the vector $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is $\overrightarrow{PR} = t\mathbf{d}$ or equivalently $\overrightarrow{OR} = \overrightarrow{OP} + t\mathbf{d}$.

In coordinate form, the equations are

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct,$$

where $R = (x, y, z)$ is the typical point on l and the parameter t takes on all real values.

For lines in the xy plane, the z component is not present; otherwise, the results are the same.

Example 5 (a) Find the equations of the line in space through the point $(3, -1, 2)$ in the direction $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.
(b) Find the equation of the line in the plane through the point $(1, -6)$ in the direction of $5\mathbf{i} - \pi\mathbf{j}$.
(c) In what direction does the line $x = -3t + 2$, $y = -2(t - 1)$, $z = 8t + 2$ point?

Solution (a) Here $P = (3, -1, 2) = (x_0, y_0, z_0)$ and $\mathbf{d} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, so $a = 2$, $b = -3$, and $c = 4$. Thus the equations are

$$x = 3 + 2t, \qquad y = -1 - 3t, \qquad z = 2 + 4t.$$

(b) Here $P = (1, -6)$ and $\mathbf{d} = 5\mathbf{i} - \pi\mathbf{j}$, so the line is

$$R = (1, -6) + (5t, -\pi t) = (1 + 5t, -6 - \pi t)$$

or

$$x = 1 + 5t, \qquad y = -6 - \pi t.$$

(c) Using the preceding box, we construct the direction $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ from the coefficients of t: $a = -3$, $b = -2$, $c = 8$. Thus the line points in the direction of $\mathbf{d} = -3\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$. ▲

Example 6 (a) Do the lines $R_1 = (t, -6t + 1, 2t - 8)$ and $R_2 = (3t + 1, 2t, 0)$ intersect?
(b) Find the "equation" of the line segment between $(1, 1, 1)$ and $(2, 1, 2)$.

Solution (a) If the lines intersect, there must be numbers t_1 and t_2 such that the corresponding points are equal: $(t_1, -6t_1 + 1, 2t_1 - 8) = (3t_2 + 1, 2t_2, 0)$; that is,

$$t_1 = 3t_2 + 1,$$
$$-6t_1 + 1 = 2t_2,$$
$$2t_1 - 8 = 0.$$

From the third equation we have $t_1 = 4$. The first equation then becomes $4 = 3t_2 + 1$ or $t_2 = 1$. We must check whether these values satisfy the middle equation:

$$-6t_1 + 1 \stackrel{?}{=} 2t_2, \quad \text{i.e.,}$$

$$-6 \cdot 4 + 1 \stackrel{?}{=} 2 \cdot 1, \quad \text{i.e.,}$$

$$-24 + 1 \stackrel{?}{=} 2.$$

The answer is no; the lines do not intersect.
(b) The *line* through $(1,1,1)$ and $(2,1,2)$ is described in parametric form by $R = (1-t)(1,1,1) + t(2,1,2) = (1+t, 1, 1+t)$, as t takes on all real values. The point R lies between $(1,1,1)$ and $(2,1,2)$ only when $0 \leqslant t \leqslant 1$, so the line *segment* is described by $R = (1+t, 1, 1+t)$, $0 \leqslant t \leqslant 1$. ▲

Since all the line segments representing a given vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ have the same length, we may define the *length* of $\mathbf{v}$ to be the length of any of these segments. To calculate the length of $\mathbf{v}$, it is convenient to use the segment $\overrightarrow{OP}$, where $P = (a,b,c)$, so that the length of $\mathbf{v}$ is just the distance from $(0,0,0)$ to (a,b,c). We apply the Pythagorean theorem twice to calculate this distance. (See Fig. 13.3.6.) Let $Q = (a,b,0)$ and $R = (a,0,0)$. Then $|OR| = |a|$ and $|RQ| = |b|$, so $|OQ| = \sqrt{a^2+b^2}$. Now $|QP| = |c|$, so applying Pythagoras' theorem again, this time to the right triangle OQP, we obtain $|OP| = \sqrt{a^2+b^2+c^2}$. We denote the length of a vector $\mathbf{v}$ by $\|\mathbf{v}\|$; it is sometimes called the *magnitude* of $\mathbf{v}$ as well.

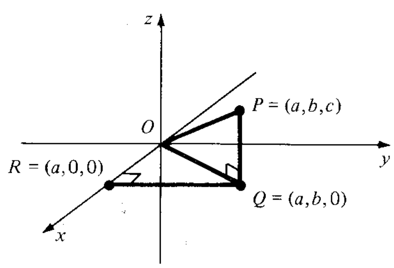


Figure 13.3.6. $|OP| = \sqrt{a^2+b^2+c^2}$.

Length of a Vector

The length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is the square root of the sum of the squares of the components of $\mathbf{v}$:

$$\|a\mathbf{i} + b\mathbf{j} + c\mathbf{k}\| = \sqrt{a^2+b^2+c^2}.$$

Example 7 (a) Find the length of $\mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$. (b) Find the values of c for which $\|\mathbf{i} + \mathbf{j} + c\mathbf{k}\| = 4$.

Solution (a) $\|\mathbf{v}\| = \sqrt{2^2 + (-6)^2 + 7^2} = \sqrt{4+36+49} = \sqrt{89} \approx 9.434$.

(b) We have $\|\mathbf{i} + \mathbf{j} + c\mathbf{k}\| = \sqrt{1+1+c^2} = \sqrt{2+c^2}$. This equals 4 when $2 + c^2 = 16$, or $c = \pm\sqrt{14}$. ▲

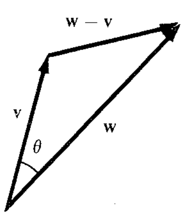


Figure 13.3.7. The law of cosines applied to vectors.

Some basic properties of length may be deduced from the law of cosines (see Section 5.1 for its proof). In terms of vectors, the law of cosines states that

$$\|\mathbf{w} - \mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, $0 \leqslant \theta \leqslant \pi$. (See Fig. 13.3.7.)

In particular, since $\cos\theta \leqslant 1$, we get

$$\begin{aligned}\|\mathbf{w}-\mathbf{v}\|^2 &= \|\mathbf{w}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta \\ &\geqslant \|\mathbf{w}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \\ &= (\|\mathbf{w}\| - \|\mathbf{v}\|)^2.\end{aligned}$$

Taking square roots and remembering that $\sqrt{x^2} = |x|$, we get

$$\|\mathbf{w}-\mathbf{v}\| \geqslant |\|\mathbf{w}\| - \|\mathbf{v}\||.$$

Hence

$$-(\|\mathbf{w}-\mathbf{v}\|) \leqslant \|\mathbf{w}\| - \|\mathbf{v}\| \leqslant \|\mathbf{w}-\mathbf{v}\|.$$

In particular, from the right-hand inequality, we get

$$\|\mathbf{w}\| \leqslant \|\mathbf{w}-\mathbf{v}\| + \|\mathbf{v}\|.$$

That is, the length of one side of a triangle is less than or equal to the sum of the lengths of the other sides. If we write $\mathbf{u} = \mathbf{w} - \mathbf{v}$, then $\mathbf{w} = \mathbf{u} + \mathbf{v}$, and the inequality above takes the useful form

$$\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|,$$

which is called the *triangle inequality*.

The relation between length and scalar multiplication is given by

$$\|r\mathbf{v}\| = |r|\,\|\mathbf{v}\|$$

since, if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\|r\mathbf{v}\| = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \sqrt{r^2}\,\sqrt{a^2+b^2+c^2} = |r|\|\mathbf{v}\|.$$

Properties of Length

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r is any number:

(1) $\|\mathbf{u}\| \geqslant 0$;
(2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$;
(3) $\|r\mathbf{u}\| = |r|\,\|\mathbf{u}\|$;

$$\left.\begin{aligned}&(4)\ \ \|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|;\\ &(4')\ \ \|\mathbf{w}-\mathbf{v}\| \geqslant |\|\mathbf{w}\| - \|\mathbf{v}\||.\end{aligned}\right\}\ \text{Triangle inequality}$$

Example 8 (a) Verify the triangle inequality (4) for $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.
(b) Prove that $\|\mathbf{u}-\mathbf{v}\| \leqslant \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Illustrate with a figure in which $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are drawn with the same base point, that is, the same "tail."

Solution (a) We have $\mathbf{u} + \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, so $\|\mathbf{u}+\mathbf{v}\| = \sqrt{9+4+1} = \sqrt{14}$. On the other hand, $\|\mathbf{u}\| = \sqrt{2}$ and $\|\mathbf{v}\| = \sqrt{6}$, so the triangle inequality asserts that $\sqrt{14} \leqslant \sqrt{2} + \sqrt{6}$. This is indeed true, since $\sqrt{14} \approx 3.74$, while $\sqrt{2} + \sqrt{6} \approx 1.41 + 2.45 = 3.86$.
(b) We find that $\mathbf{u} - \mathbf{v} = (\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})$, so the result follows from the triangle inequality with $\mathbf{u}$ replaced by $\mathbf{u} - \mathbf{w}$ and $\mathbf{v}$ replaced by $\mathbf{w} - \mathbf{v}$. Geometrically, we are considering the shaded triangle in Fig. 13.3.8. ▲

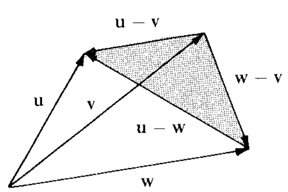


Figure 13.3.8. Illustrating the inequality $\|\mathbf{u}-\mathbf{v}\| \leqslant \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$.

The length of a vector can have interpretations other than the geometric one given above. For example, suppose that an object is moving uniformly along a straight line. What physical quantity is represented by the length of its velocity vector? To answer this, let $\mathbf{v}$ be the velocity vector. The displacement vector from its position P at any time to its position Q, t units of time later, is $t\mathbf{v}$. The

distance between P and Q is then $|t|\|\mathbf{v}\|$, so the length $\|\mathbf{v}\|$ of the velocity vector represents the ratio of distance travelled to elapsed time—it is called the *speed*.

A vector $\mathbf{u}$ is called a *unit vector* if its length is equal to 1. If $\mathbf{v}$ is any nonzero vector, $\|\mathbf{v}\| \neq 0$ then we can obtain a unit vector pointing in the direction of $\mathbf{v}$ by taking $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$. In fact,

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

We call $\mathbf{u}$ the *normalization* of $\mathbf{v}$.

Example 9 (a) Normalize $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \frac{1}{2}\mathbf{k}$. (b) Find unit vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in the plane such that $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

Solution (a) We have $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 1/2^2} = (1/2)\sqrt{53}$, so the normalization of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{4}{\sqrt{53}}\mathbf{i} + \frac{6}{\sqrt{53}}\mathbf{j} - \frac{1}{\sqrt{53}}\mathbf{k}.$$

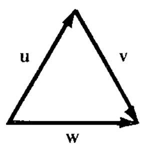


Figure 13.3.9. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are represented by the sides of an equilateral traingle.

(b) A triangle whose sides represent $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ must be equilateral as in Fig. 13.3.9. Knowing this, we may take $\mathbf{w} = \mathbf{i}$, $\mathbf{u} = \frac{1}{2}\mathbf{i} + (\sqrt{3}/2)\mathbf{j}$, $\mathbf{v} = \frac{1}{2}\mathbf{i} - (\sqrt{3}/2)\mathbf{j}$. Check that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and that $\mathbf{u} + \mathbf{v} = \mathbf{w}$. ▲

Finally, we can use the formula for the length of vectors to obtain a formula for the distance between any two points in space. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, then the distance between P_1 and P_2 is the length of the vector from P_2 to P_1; that is,

$$\begin{aligned} |P_1P_2| &= \|(x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k}\| \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\,. \end{aligned}$$

Distance Formula

If P_1 has coordinates (x_1, y_1, z_1) and P_2 has coordinates (x_2, y_2, z_2), then the distance between P_1 and P_2 is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\,.$$

Example 10 (a) Find the distance between $(2, 1, 0)$ and $(3, -2, 6)$.
(b) Let $P_t = t(1, 1, 1)$.
(i) What is the distance from P_t to $(3, 0, 0)$?
(ii) For what value of t is the distance shortest?
(iii) What is the shortest distance?

Solution (a) The distance is

$$\begin{aligned} \sqrt{(2-3)^2 + [1-(-2)]^2 + (0-6)^2} &= \sqrt{(-1)^2 + 3^2 + (-6)^2} \\ &= \sqrt{1 + 9 + 36} = \sqrt{46}\,. \end{aligned}$$

(b) (i) By the distance formula, the distance is

$$\sqrt{(t-3)^2+(t-0)^2+(t-0)^2}$$
$$=\sqrt{t^2-6t+9+t^2+t^2}=\sqrt{3t^2-6t+9}\,.$$

(ii) The distance is shortest when $3t^2-6t+9$ is least—that is, when $(d/dt)(3t^2-6t+9)=6t-6=0$, or $t=1$.

(iii) For $t=1$, the distance in (i) is $\sqrt{6}$. ▲

Exercises for Section 13.3

Use vector methods in Exercises 1–6.

1. Show that the line segment joining the midpoints of two sides of a triangle is parallel to and has half of the length of the third side.
2. Prove that the medians of the triangle intersect in a point two-thirds of the way along any median from a vertex to the midpoint of the opposite side.
3. Prove that if PQR is a triangle in space and $b>0$ is a number, then there is a triangle with sides parallel to those of PQR and side lengths b times those of PQR.
4. Prove that if the corresponding sides of two triangles are parallel, then the lengths of corresponding sides have a common ratio. (Assume that the triangles are not degenerated into lines.)
5. Find the point in the plane two-thirds of the way from the origin to the midpoint of the line segment between $(1,1)$ and $(2,-2)$.
6. Let $P=(3,5,2)$ and $Q=(2,5,3)$. Find the point R such that Q is the midpoint of the line segment PR.

Write equations for the lines in Exercises 7–10.

7. The line through $(1,1,0)$ and $(0,0,1)$.
8. The line through $(2,0,0)$ and $(0,1,0)$.
9. The line through $(0,0,0)$ and $(1,1,1)$.
10. The line through $(-1,-1,0)$ and $(1,8,-4)$.

Write parametric equations for the lines in Exercises 11–14.

11. The line through the point $(1,1,0)$ in the direction of vector $-\mathbf{i}-\mathbf{j}+\mathbf{k}$.
12. The line through $(0,1,0)$ in the direction $\mathbf{j}$.
13. The line in the plane through $(-1,-2)$ and in direction $3\mathbf{i}-2\mathbf{j}$.
14. The line in the plane through $(2,-1)$ and in direction $-\mathbf{i}-\mathbf{j}$.

15. At what point does the line through $(0,1,2)$ with direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$ cross the xy plane?
16. Where does the line through $(3,4,5)$ and $(6,7,8)$ meet the yz plane?
17. Do the lines given by $R_1=(t,3t-1,4t)$ and $R_2=(3t,5,1-t)$ intersect?
18. Find the unique value of c for which the lines $R_1=(t,-6t+c,2t-8)$ and $R_2=(3t+1,2t,0)$ intersect.

Compute the length of the vectors in Exercises 19–24.

19. $\mathbf{i}+\mathbf{j}+\mathbf{k}$
20. $2\mathbf{i}+\mathbf{j}$
21. $\mathbf{i}+\mathbf{k}$
22. $3\mathbf{i}+4\mathbf{j}$
23. $2\mathbf{i}+2\mathbf{k}$
24. $\mathbf{i}-\mathbf{j}-3\mathbf{k}$
25. For what a is $\|a\mathbf{i}-3\mathbf{j}+\mathbf{k}\|=16$?
26. For what b is $\|\mathbf{i}-b\mathbf{j}+2\mathbf{k}\|=3$?
27. Verify the triangle inequality for the vectors in Exercises 19 and 21.
28. Verify the triangle inequality for the vectors in Exercises 20 and 22.
29. Find unit vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in the plane such that $\mathbf{u}+2\mathbf{v}+\mathbf{w}=\mathbf{0}$.
30. Find unit vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ in the plane such that $\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{z}=\mathbf{0}$.
31. Show that $(0,0,1)$ $(0,1,0)$, and $(1,0,0)$ are the vertices of an equilateral triangle. How long is each side?
32. Find an equilateral triangle in space which shares just one side with the one in Exercise 31.
33. Normalize the vectors in Exercises 19 and 21.
34. Normalize the vectors in Exercises 20 and 22.

Find the distance between the pairs of points in Exercises 35–38.

35. $(1,1,3)$ and $(2,2,2)$
36. $(2,0,0)$ and $(2,1,2)$
37. $(1,1,2)$ and $(1,2,3)$
38. $(1,2,3)$ and $(3,2,1)$
39. Let $P_t=t(3,2,1)$.
    (i) What is the distance from P_t to $(2,0,0)$?
    (ii) For what value of t is the distance shortest?
    (iii) What is that shortest distance?
40. Draw a figure, similar to Fig. 13.3.6, to illustrate the distance formula on p. 666.
41. A boat whose top speed in still water is 12 knots points north and steams at full power. If there is an eastward current of 5 knots, what is the speed of the boat?
42. A ship starting at $(0,0)$ proceeds at a speed of 10 knots directly toward a buoy located at $(3,4)$. (The chart is measured in nautical miles; a knot equals 1 nautical mile per hour.)
    (a) What is the ship's point of closest approach to a rock located at $(2,2)$?
    (b) After how long does it reach this point?
    (c) How far is this point from the rock?

★43. Derive the point-point form of the equation of a line obtained in Section R.4 from the parametric

form obtained in this section. Comment on the case in which $x_1 = x_2$.

★44. Derive the point-direction form for the parameteric equation of a line from the point-point form. [*Hint*: If a line through P is to have direction $\mathbf{d}$, what other point must lie on the line?]

★45. When does equality hold in the triangle inequality? (You might try using the law of cosines as was done in the text.) Test your conclusion on the vectors in Exercises 21 and 23.

★46. The potential V produced at (x, y, z) by charges q_1 and q_2 of opposite sign placed at distances x_1 and x_2 from the origin along the x axis is given by

$$V = \frac{q_1}{4\pi\epsilon_0\|\mathbf{r}-\mathbf{r}_1\|} + \frac{q_2}{4\pi\epsilon_0\|\mathbf{r}-\mathbf{r}_2\|}.$$

In this formula, $\mathbf{r}$ is the vector from the origin to the point (x, y, z). The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors from the origin to the respective charges q_1 and q_2.

(a) Express the formula for V entirely in terms of the scalar quantities $x, y, z, x_1, x_2, q_1, q_2, \epsilon_0$.

(b) Show that the locus of points (x, y, z) for which $V = 0$ is a plane or a sphere whose radius is $|q_1 q_2(x_1 - x_2)/(q_1^2 - q_2^2)|$ and whose center is on the x axis or is a plane.

13.4 The Dot Product

The dot product of two unit vectors is the cosine of the angle between them.

To introduce the dot product, we will calculate the angle θ between two vectors in terms of the components of the vectors.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two vectors, we have seen (Fig. 13.3.7) that

$$\|\mathbf{v}_2 - \mathbf{v}_1\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - 2\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta,$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$; $0 \leqslant \theta \leqslant \pi$. Therefore,

$$2\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - \|\mathbf{v}_2 - \mathbf{v}_1\|^2. \tag{1}$$

If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, then the right-hand side of equation (1) is

$$\begin{aligned}&\left(a_1^2 + b_1^2 + c_1^2\right) + \left(a_2^2 + b_2^2 + c_2^2\right) - \left[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\right]\\ &\quad = 2(a_1a_2 + b_1b_2 + c_1c_2).\end{aligned}$$

Thus we have proved that

$$\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta = a_1a_2 + b_1b_2 + c_1c_2. \tag{2}$$

This very convenient formula enables us to compute $\cos\theta$ and hence θ; thus the quantity on the right-hand side deserves a special name. If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors, the number $a_1a_2 + b_1b_2 + c_1c_2$ is called their *dot product* and is denoted by $\mathbf{v}_1 \cdot \mathbf{v}_2$. The dot product in the plane is defined analogously; just think of c_1 and c_2 as being zero.

Notice that the dot product of two vectors is a number, not a vector. It is sometimes called the *scalar product* (do not confuse this with scalar *multiplication*) or the *inner product*.

Example 1 (a) If $\mathbf{v}_1 = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}_2 = \mathbf{i} - \mathbf{j} + \mathbf{k}$, calculate $\mathbf{v}_1 \cdot \mathbf{v}_2$.
(b) Calculate $(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (3\mathbf{k} - 2\mathbf{j})$.

Solution (a) $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 \cdot 1 + 1 \cdot (-1) + (-2) \cdot 1 = 3 - 1 - 2 = 0$.
(b) $(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (3\mathbf{k} - 2\mathbf{j}) = (2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (0\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = 2 \cdot 0 - 1 \cdot 2 - 1 \cdot 3 = -5$. ▲

Combining formula (2) with the definition of the dot product gives

$$\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta = \mathbf{v}_1 \cdot \mathbf{v}_2, \tag{3}$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.

We may solve (3) for $\cos\theta$ to obtain the formula

$$\cos\theta = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|};$$

i.e., $$\theta = \cos^{-1}\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|}\right) \text{if} \quad \mathbf{v}_1 \neq \mathbf{0} \quad \text{and} \quad \mathbf{v}_2 \neq \mathbf{0}.$$

Example 2 (a) Find the angle between the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}-\mathbf{k}$. (b) Find the angle between $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

Solution (a) Let $\mathbf{v}_1 = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v}_2 = \mathbf{i}+\mathbf{j}-\mathbf{k}$. Then $\|\mathbf{v}_1\| = \sqrt{3}$, $\|\mathbf{v}_2\| = \sqrt{3}$, and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 1\cdot 1 + 1\cdot 1 - 1\cdot 1 = 1$. Hence $\cos\theta = \frac{1}{3}$, so $\theta = \cos^{-1}(\frac{1}{3}) \approx 1.23$ radians (70°32′).

(b) From Example 1(a), $(3\mathbf{i}+\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k}) = 0$, so $\cos\theta = 0$ and hence $\theta = \pi/2$. ▲

From (3) we get

$$|\mathbf{v}_1 \cdot \mathbf{v}_2| = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\,|\cos\theta|.$$

However, $|\cos\theta| \leqslant 1$, so we have

$$|\mathbf{v}_1 \cdot \mathbf{v}_2| \leqslant \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|.$$

This is a useful inequality called the Schwarz inequality (and sometimes the Cauchy–Schwarz–Buniakowski inequality).

From either (2) or (3), we notice that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\mathbf{v}\cdot\mathbf{v} = a^2 + b^2 + c^2 = \|\mathbf{v}\|^2.$$

Since two nonzero vectors are perpendicular when $\theta = \pi/2$—that is, when $\cos\theta = 0$—we have an algebraic test for perpendicularity: the nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular when $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. (We adopt the convention that the zero vector is perpendicular to every vector.) The synonyms "orthogonal" or "normal" are also used for "perpendicular."

The Dot Product

Algebraic definition:

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})\cdot(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = a_1a_2 + b_1b_2 + c_1c_2.$$

Geometric interpretation:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta,$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, $0 \leqslant \theta \leqslant \pi$. In particular,

$$\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}.$$

Properties:

1. $\mathbf{u}\cdot\mathbf{u} \geqslant 0$ for any vector $\mathbf{u}$.
2. $\mathbf{u}\cdot\mathbf{u} = 0$ only if $\mathbf{u} = \mathbf{0}$.
3. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.
4. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$.
5. $(a\mathbf{u})\cdot\mathbf{v} = a(\mathbf{u}\cdot\mathbf{v})$.
6. $|\mathbf{u}\cdot\mathbf{v}| \leqslant \|\mathbf{u}\|\,\|\mathbf{v}\|$ (Schwarz inequality).
7. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular when $\mathbf{u}\cdot\mathbf{v} = 0$.

Example 3 Find a unit vector in the plane which is orthogonal to $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

Solution If $\mathbf{w} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to $\mathbf{i} - 3\mathbf{j}$, we must have $0 = \mathbf{v} \cdot \mathbf{w} = a - 3b$; that is, $a = 3b$. If $\mathbf{w}$ is to be a unit vector, we must also have $1 = a^2 + b^2 = (3b)^2 + b^2 = 10b^2$, so $b = \pm 1/\sqrt{10}$ and $a = \pm 3/\sqrt{10}$. Thus there are two possible solutions: $(\pm 1/\sqrt{10})(3\mathbf{i} + \mathbf{j})$. (See Fig. 13.4.1.) ▲

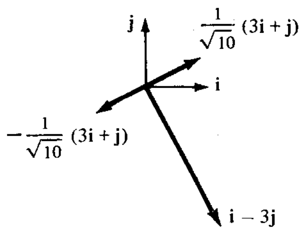


Figure 13.4.1. There are two unit vectors orthogonal to $\mathbf{i} - 3\mathbf{j}$.

Example 4 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the plane; assume that $\mathbf{u}$ is nonzero.

(a) Show that $\mathbf{w} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{u}/\mathbf{u} \cdot \mathbf{u})\mathbf{u}$ is orthogonal to $\mathbf{u}$.
(b) Sketch the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} - \mathbf{w}$. The vector $\mathbf{v} - \mathbf{w} = (\mathbf{v} \cdot \mathbf{u}/\mathbf{u} \cdot \mathbf{u})\mathbf{u}$ is called the *orthogonal projection* of $\mathbf{v}$ on $\mathbf{u}$. Why?
(c) Find the orthogonal projection of $\mathbf{i} + \mathbf{j}$ on $\mathbf{i} - 2\mathbf{j}$.

Solution (a) We compute

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot \left(\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}\right).$$

By the algebraic properties of the dot product, this is equal to

$$\mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0,$$

so $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{w}$ is orthogonal to $\mathbf{u}$.

(b) We note that $\mathbf{v} - \mathbf{w} = (\mathbf{v} \cdot \mathbf{u}/\mathbf{u} \cdot \mathbf{u})\mathbf{u}$ is a multiple of $\mathbf{u}$. Thus the configuration of vectors must be as in Fig. 13.4.2. The vector $\mathbf{v} - \mathbf{w}$ is called the *orthogonal projection* of $\mathbf{v}$ on $\mathbf{u}$ because it is obtained by dropping a perpendicular from the "tip" of $\mathbf{v}$ to the line determined by $\mathbf{u}$. (The base points of $\mathbf{u}$ and $\mathbf{v}$ must be the same for this construction.)

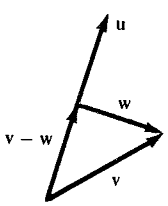


Figure 13.4.2. The vector $\mathbf{v} - \mathbf{w} = (\mathbf{v} \cdot \mathbf{u}/\mathbf{u} \cdot \mathbf{u})\mathbf{u}$ is the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$.

(c) With $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$, the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$ is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{1-2}{1+4}(\mathbf{i} - 2\mathbf{j}) = -\frac{1}{5}(\mathbf{i} - 2\mathbf{j}).$$

(See Fig. 13.4.3.) ▲

We can use the dot product to find the distance from a point $Q = (x_1, y_1, z_1)$ to the line l which passes through a point $P = (x_0, y_0, z_0)$ and has the direction $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Indeed, in Fig. 13.4.4 the distance from Q to the line is the

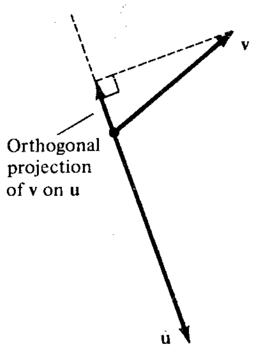


Figure 13.4.3. The orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$ equals $-\frac{1}{5}\mathbf{u}$.

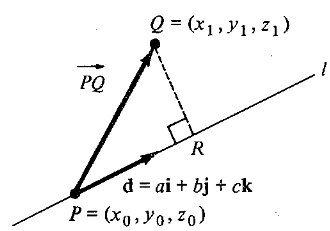


Figure 13.4.4. $\|\overrightarrow{QR}\|$ is the distance from Q to l.

distance between Q and R, where R is chosen on l in such a way that $\overrightarrow{PR}$ and $\overrightarrow{QR}$ are orthogonal. Then $\overrightarrow{PR}$ is the orthogonal projection of $\overrightarrow{PQ}$ on the line l. Thus, by Example 4,

$$\overrightarrow{PR} = \frac{\overrightarrow{PQ}\cdot\mathbf{d}}{\mathbf{d}\cdot\mathbf{d}}\mathbf{d} = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{a^2 + b^2 + c^2}\mathbf{d}.$$

By Pythagoras' theorem, $\|\overrightarrow{RQ}\| = \sqrt{\|\overrightarrow{PQ}\|^2 - \|\overrightarrow{PR}\|^2}$ which gives

$$\operatorname{dist}(Q,l) = \left\{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - \frac{\left[a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)\right]^2}{a^2 + b^2 + c^2}\right\}^{1/2} \tag{4}$$

as the distance from Q to the line l.

Example 5 Find the distance from $(1,1,2)$ to the line through $(2,0,0)$ in the direction $(1/\sqrt{2})\mathbf{i} - (1/\sqrt{2})\mathbf{j}$.

Solution In formula (4), we set $(x_1, y_1, z_1) = (1,1,2)$, $(x_0, y_0, z_0) = (2,0,0)$ and obtain a, b, c from $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (1/\sqrt{2})\mathbf{i} - (1/\sqrt{2})\mathbf{j}$ to be $a = 1/\sqrt{2}$, $b = -1\sqrt{2}$, $c = 0$. Thus,

$$\operatorname{dist}(Q,l) = \left\{(1-2)^2 + 1^2 + 2^2 - \frac{\left[\frac{1}{\sqrt{2}}(1-2) - \frac{1}{\sqrt{2}}1\right]^2}{\frac{1}{2} + \frac{1}{2}}\right\}^{1/2}$$

$$= (6-2)^{1/2} = 2. \blacktriangle$$

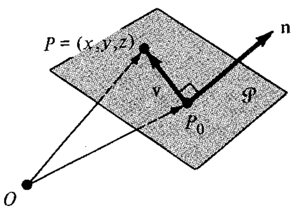


Figure 13.4.5. The plane $\mathscr{P}$ is perpendicular to the vector **n**.

The dot product makes it simple to determine the equation of a plane. Suppose that a plane $\mathscr{P}$ passes through a point $P_0 = (x_0, y_0, z_0)$ and is perpendicular to a vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. (See Fig. 13.4.5.)

Let $P = (x, y, z)$ be a point on $\mathscr{P}$. Then $\mathbf{n}$ must be perpendicular to the vector $\mathbf{v}$ from P_0 to P; that is, $\mathbf{n}\cdot\mathbf{v} = 0$, or

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k})\cdot\left[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}\right] = 0.$$

Hence

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

We call $\mathbf{n}$ the *normal vector* of the plane. If we let $D = -(Ax_0 + By_0 + Cz_0)$, the equation of the plane becomes

$$Ax + By + Cz + D = 0.$$

Example 6 Find the equation of the plane through $(1,1,1)$ with normal vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution Here $A\mathbf{i} + B\mathbf{j} + C\mathbf{k} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and $(x_0, y_0, z_0) = (1,1,1)$, so we get

$$A(x-1) + B(y-1) + C(z-1) = 0.$$

Hence

$$2(x-1) + (y-1) - 2(z-1) = 0,$$

$$2x + y - 2z = 1. \blacktriangle$$

Example 7 Find a unit normal vector to the plane $3x + y - z = 10$. Sketch the plane.

Solution A normal vector is obtained by making a vector out of the coefficients of x, y, and z; that is, $(3, 1, -1)$. Normalizing, we get

$$(3,1,-1)/\sqrt{9+1+1} = (3/\sqrt{11}\,, 1/\sqrt{11}\,, -1/\sqrt{11}); \text{ i.e. } \frac{1}{\sqrt{11}}(3\mathbf{i}+\mathbf{j}-\mathbf{k}).$$

We may sketch the plane by noting where it meets the coordinate axes. For example, setting $y = z = 0$, we see that $(\frac{10}{3}, 0, 0)$ lies on the plane (see Fig. 13.4.6). ▲

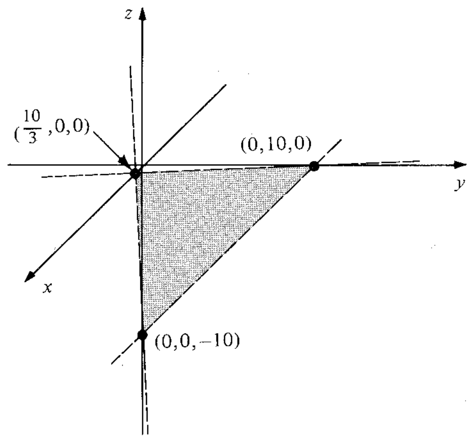


Figure 13.4.6. The plane $3x + y - z = 10$.

Equation of a Plane in Space

The equation of the plane through (x_0, y_0, z_0) with normal vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{5}$$

or

$$Ax + By + Cz + D = 0. \tag{6}$$

Example 8 (a) Find the equation of the plane passing through the point $(3, -1, -1)$ and perpendicular to the vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. (b) Find the equation of the plane containing the points $(1, 1, 1)$, $(2, 0, 0)$, and $(1, 1, 0)$.

Solution (a) We use the first displayed equation (5) in the preceding box, choosing the point $(x_0, y_0, z_0) = (3, -1, -1)$ and components of the normal vector to be $A = 1$, $B = -2$, $C = 1$ to give

$$1(x-3) - 2(y+1) + 1(z+1) = 0$$

which simplifies to $x - 2y + z = 4$.

(b) The general equation of a plane has the form (6) $Ax + By + Cz + D = 0$. Since the points $(1, 1, 1)$, $(2, 0, 0)$, and $(1, 1, 0)$ lie on the plane, the coefficients A, B, C, D satisfy the three equations:

$$\begin{aligned} A+B+C+D &= 0,\\ 2A \qquad\quad +D &= 0,\\ A+B \quad\ \ +D &= 0. \end{aligned}$$

Proceeding by elimination, we reduce this system to the form

$$2A + D = 0 \qquad \text{(second equation)},$$
$$2B + D = 0 \qquad \text{(twice the third equation minus the second)},$$
$$C = 0 \qquad \text{(first equation minus the third)}.$$

Since the numbers A, B, C, and D are determined only up to a common factor, we can fix the value of one of them and then the others will be determined uniquely. If we let $D = -2$, then $A = 1$, $B = 1$, $C = 0$. Thus $x + y - 2 = 0$ is an equation of the plane that contains the given points. (You may go back and verify that the given points actually satisfy this equation.) ▲

Example 9 Where does the line through the origin in the direction of $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ meet the plane $x + y + 2z = 5$? Use your answer to find the distance from the origin to this plane. Sketch.

Solution The line has parametric equations $x = t, y = t, z = 2t$. It meets the plane when $x + y + 2z = t + t + 4t = 5$; that is, when $t = \frac{5}{6}$. The point of intersection is $P_1 = (\frac{5}{6}, \frac{5}{6}, \frac{5}{3})$.

Since a normal to the plane is $\mathbf{n} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, which is the same as the direction vector of this line, we see that the line is perpendicular to the plane at P_1. If P is also in the plane, consideration of the right triangle OP_1P shows that $\overrightarrow{OP_1}$ must be shorter than $\overrightarrow{OP}$ (see Fig. 13.4.7). Thus the distance from the origin to the plane is the length of $\overrightarrow{OP_1}$:

$$\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}} = \frac{\sqrt{150}}{6} = \frac{5\sqrt{6}}{6}. \ ▲$$

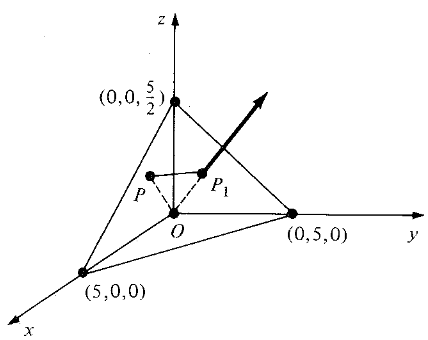


Figure 13.4.7. The vector from O to the closest point P_1 on a plane is perpendicular to the plane.

Let $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ be the equation of a plane $\mathscr{P}$ through the point $P = (x_0, y_0, z_0)$ in space. Let us use the basic ideas of the preceding example to determine the distance from a point $Q = (x_1, y_1, z_1)$ to the plane (see Figure 13.4.8). Consider the vector

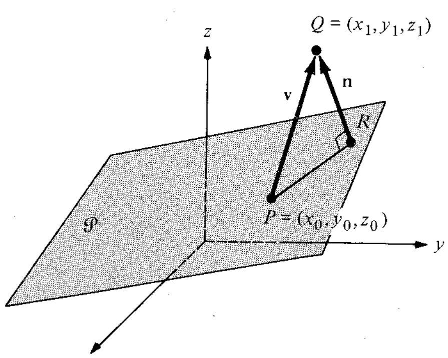


Figure 13.4.8. The geometry for determining the distance from a point to a plane.

$$\mathbf{n} = \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}},$$

which is a unit vector normal to the plane. Next drop a perpendicular from Q to the plane and construct the triangle PQR shown in Figure 13.4.8. The distance $d = \overrightarrow{RQ}$ is the length of the projection of $\mathbf{v} = \overrightarrow{PQ}$ (the vector from P to Q) onto $\mathbf{n}$; thus

$$\begin{aligned}\text{distance} &= |\mathbf{v}\cdot\mathbf{n}| = |[(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}]\cdot\mathbf{n}| \\ &= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}}.\end{aligned}$$

If the plane is given in the form $Ax + By + Cz + D = 0$, choose a point (x_0, y_0, z_0) on it and note that $D = -(Ax_0 + By_0 + Cz_0)$. Substituting in the previous formula gives

$$\operatorname{dist}(Q, \mathscr{P}) = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{7}$$

for the distance from Q to $\mathscr{P}$.

Example 10 Find the distance from $Q = (2, 0, -1)$ to the plane $\mathscr{P}$: $3x - 2y + 8z + 1 = 0$.

Solution We substitute into (7) the values $x_1 = 2$, $y_1 = 0$, $z_1 = -1$ (from the point) and $A = 3$, $B = -2$, $C = 8$, $D = 1$ (from the plane) to give

$$\operatorname{dist}(Q, \mathscr{P}) = \frac{|3\cdot 2 + (-2)\cdot 0 + 8(-1) + 1|}{\sqrt{3^2 + (-2)^2 + 8^2}} = \frac{|-1|}{\sqrt{77}} = \frac{1}{\sqrt{77}}. \ \blacktriangle$$

Exercises for Section 13.4

Compute the dot products in Exercises 1–4.

1. $(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
2. $(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot(\mathbf{i} + \mathbf{k})$
3. $\mathbf{i}\cdot\mathbf{j}$
4. $(3\mathbf{i} + 4\mathbf{j})\cdot(3\mathbf{j} + 4\mathbf{k})$
5. Find the angle between the pair of vectors in Exercise 1.
6. Find the angle between the pair of vectors in Exercise 2.
7. Find the angle between the pair of vectors in Exercise 3.
8. Find the angle between the pair of vectors in Exercise 4.
9. Find a unit vector in the xy plane which is orthogonal to $2\mathbf{i} - \mathbf{j}$.
10. Find a unit vector in the xy plane which is orthogonal to $3\mathbf{j} - 5\mathbf{i}$.
11. Use the formula $(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot\mathbf{i} = 1$ to find the angle between the diagonal of a cube and one of its edges. Sketch.
12. (a) Show that if $\|\mathbf{u}\| = \|\mathbf{v}\|$, and $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are perpendicular. (b) Use the result of part (a) to prove that any triangle inscribed in a circle, with one side of the triangle as a diameter, is a right triangle.
13. Show that the length of the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$ is equal to $\|\mathbf{v}\|\,|\cos\theta|$, where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$.
14. Use vector methods to prove that a triangle is isosceles if and only if its base angles are equal.
15. Find the distance from $(2, 8, -1)$ to the line through $(1, 1, 1)$ in the direction of the vector $(1/\sqrt{3})\mathbf{i} + (1/\sqrt{3})\mathbf{j} + (1/\sqrt{3})\mathbf{k}$.
16. Find the distance from $(1, 1, -1)$ to the line through $(2, -1, 2)$ in the direction of $\mathbf{k}$.
17. Find the distance from $(1, 1, 2)$ to the line $x = 3t + 2$, $y = -t - 1$, $z = t + 1$.
18. Find the distance from $(1, 1, 0)$ to the line through $(1, 0, -1)$ and $(2, 3, 1)$.

Give the equation for each of the planes in Exercises 19–24.

19. The plane through the origin orthogonal to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
20. The plane through $(1, 0, 0)$ orthogonal to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

21. The plane through the origin orthogonal to **i**.
22. The plane containing (a, b, c) with normal vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
23. The plane containing the points $(0,0,1)$, $(1,1,1)$ and $(0,1,0)$.
24. The plane containing the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

Find a unit vector orthogonal to each of the planes in Exercises 25–28.

25. The plane given by $2x + 3y + z = 0$.
26. The plane given by $8x - y - 2z + 10 = 0$.
27. The plane through the origin containing the points $(1,1,1)$ and $(1,1,-1)$.
28. The plane containing the line $(1 + t, 1 - t, t)$ and the point $(1,1,1)$.

Find the equation of the objects in Exercises 29–32.

29. The plane containing $(0,1,0)$, $(1,0,0)$, $(0,0,1)$.
30. The line through $(1,2,1)$ and $(-1,1,0)$.
31. The line through $(1,1,1)$ and orthogonal to the plane in Exercise 29.
32. The line through $(0,0,0)$ which passes through and is orthogonal to the line in Exercise 30.

33. Where does the line through the origin in the direction of $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ meet the plane $2x - y + 3z = 7$? Find the distance from the origin to this plane.
34. Where does the line through $P = (1,1,1)$ in the direction of $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ meet the plane $2x - y + 3z = 7$? Find the distance from P to this plane.
35. The planes $3x + 4y + 5z = 6$ and $x - y + z = 4$ meet in a line. Find the parametric equations of this line.
36. Find the parametric equations of the line where the plane $x + y = z$ meets the plane $y + z = x$.
37. Find the distance from the point $(1,1,1)$ to the plane $x - y - z + 10 = 0$.
38. Find the distance from the point $(2,-1,2)$ to the plane $2x - y + z = 5$.
39. Find the distance from the origin to the plane through $(1,2,3)$, $(-1,2,3)$, and $(0,0,1)$.
40. Find the distance from the point $(4,2,0)$ to the plane through $(0,0,0)$, $(1,1,1)$, and $(1,1,2)$.
41. Show that the locus of points in the plane equidistant from two given points is a line, and give an equation for that line in terms of the coordinates of the two points.
42. Use vector methods to show that if three parallel lines in the plane cut off equal segments on one transversal, then they do so on any transversal.
43. Compute the following:
    (a) $(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{j} - \mathbf{k})$.
    (b) $[(3\mathbf{i} - \mathbf{j} - \mathbf{k}) - (\mathbf{i} + \mathbf{j})] \cdot \mathbf{j}$.
    (c) The distance between $(1,0,2)$ and $(3,2,4)$.
    (d) The length of $(\mathbf{i} - \mathbf{j} - \mathbf{k}) + (2\mathbf{j} - \mathbf{k} + \mathbf{i})$.
44. Find the following:
    (a) A unit normal to the plane $x - 2y + z = 0$.
    (b) A vector orthogonal to the vectors $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
    (c) The angle between $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{k} - \mathbf{i}$.
    (d) A vector in space making an angle of 45° with **i** and 60° with **j**.
45. Let P_1 and P_2 be points in the plane. Give an equation of the form $ax + by = c$ for the perpendicular bisector of the line segment between P_1 and P_2.
46. Given nonzero vectors **a** and **b**, show that the vector $\mathbf{v} = \|\mathbf{a}\|\mathbf{b} + \|\mathbf{b}\|\mathbf{a}$ bisects the angle between **a** and **b**.

Exercises 47–50 form a unit.

47. Suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular unit vectors in the plane, and let **v** be an arbitrary vector. Show that $\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2$. The numbers $\mathbf{v} \cdot \mathbf{e}_1$ and $\mathbf{v} \cdot \mathbf{e}_2$ are called the *components* of **v** in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$. This expression of **v** as a sum of vectors pointing in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$ is called the *orthogonal decomposition* of **v** relative to $\mathbf{e}_1$ and $\mathbf{e}_2$.
48. Consider the vectors $\mathbf{e}_1 = (1/\sqrt{2})(\mathbf{i} + \mathbf{j})$ and $\mathbf{e}_2 = (1/\sqrt{2})(\mathbf{i} - \mathbf{j})$ in the plane. Check that $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors perpendicular to each other and express each of the following vectors in the form $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ (that is, as a *linear combination* of $\mathbf{e}_1$ and $\mathbf{e}_2$):
    (a) $\mathbf{v} = \mathbf{i}$, (b) $\mathbf{v} = \mathbf{j}$,
    (c) $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$, (d) $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$.
49. Suppose that a force **F** (for example, gravity) is acting vertically downward on an object sitting on a plane which is inclined at an angle of 45° to the horizontal. Express this force as a sum of a force acting parallel to the plane and one acting perpendicular to it.
50. Suppose that an object moving in direction $\mathbf{i} + \mathbf{j}$ is acted on by a force given by the vector $2\mathbf{i} + \mathbf{j}$. Express this force as a sum of a force in the direction of motion and a force perpendicular to the direction of motion.

51. A force of 6 newtons makes an angle of $\pi/4$ radians with the y axis, pointing to the right. The force acts against the movement of an object along the straight line connecting $(1,2)$ to $(5,4)$.
    (a) Find a formula for the force vector **F**.
    (b) Find the angle θ between the displacement direction $\mathbf{D} = (5-1)\mathbf{i} + (4-2)\mathbf{j}$ and the force direction **F**.
    (c) The *work* done is $\mathbf{F} \cdot \mathbf{D}$, or equivalently, $\|\mathbf{F}\|\,\|\mathbf{D}\|\cos\theta$. Compute the work from both formulas and compare.
52. A fluid flows across a plane surface with uniform vector velocity **v**. Let **n** be a unit normal to the plane surface. Show that $\mathbf{v} \cdot \mathbf{n}$ is the volume of fluid that passes through a unit area of the plane in unit time.

53. Establish the following properties of the dot product:
 (a) $\mathbf{u} \cdot \mathbf{u} \geqslant 0$ for any vector $\mathbf{u}$.
 (b) If $\mathbf{u} \cdot \mathbf{u} = 0$, then $\mathbf{u} = \mathbf{0}$.
 (c) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for any vector $\mathbf{u}$ and $\mathbf{v}$.
 (d) $(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = a(\mathbf{u} \cdot \mathbf{w}) + b(\mathbf{v} \cdot \mathbf{w})$ for any numbers a and b and any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

54. Let:
 $L_1 =$ the line $(2,1,1) + t(1,1,1)$;
 $L_2 =$ the line $(1 + 7t, 7t - 2, 2 + 7t)$;
 $L_3 =$ the line $(1,0,8) + t(1,1,9)$;
 $L_4 =$ the line through the points $(-1,0,1)$ and $(1,2,19)$.
 (a) Determine whether each of the following pairs of lines is parallel or intersects. If the lines intersect, find the point of intersection.
  (i) L_1 and L_2
  (ii) L_2 and L_3
  (iii) L_1 and L_3
  (iv) L_2 and L_4
  (v) L_1 and L_4
  (vi) L_3 and L_4
 (b) For each pair of lines in part (a) which lie in a plane (that is, are not skew), find an equation for that plane.
 (c) For each of the lines L_1 to L_4, find the point of the closest approach to the origin and an equation for the plane perpendicular to that line through that point.

55. A construction worker is checking the architect's plans for some sheet metal construction. One diagram contains a triangle with sides 12.5, 16.7, 20.9, but no angles have been included. The worker gets out a calculator to check that $(12.5)^2 + (16.7)^2 \approx (20.9)^2$, then marks the angle opposite the long side as 90°.
 (a) Explain from the law of cosines the reason why the worker's actions are essentially correct.
 (b) The angle is not exactly 90°, from the data given. What percentage error is present?

★56. Let P_1 and P_2 be points in the plane with *polar* coordinates (r_1, θ_1) and (r_2, θ_2), respectively, and let $\mathbf{u}_1$ be the vector from O to P_1 and $\mathbf{u}_2$ the vector from O to P_2. Show that $\mathbf{u}_1 \cdot \mathbf{u}_2 = r_1 r_2 \cos(\theta_1 - \theta_2)$. [*Hint*: Use a trigonometric identity.]

★57. Suppose that $R = P_0 + t(a,b,c)$ is the line through P_0 in the direction $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Let $\mathbf{u} = \mathbf{d}/\|\mathbf{d}\| = (\mu, \lambda, \nu)$. Let:

 $\alpha =$ angle from $\mathbf{i}$ to $\mathbf{d}$;
 $\beta =$ angle from $\mathbf{j}$ to $\mathbf{d}$;
 $\gamma =$ angle from $\mathbf{k}$ to $\mathbf{d}$.

 These are called the *direction angles* of the line. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called its *direction cosines* (see Fig. 13.4.9).

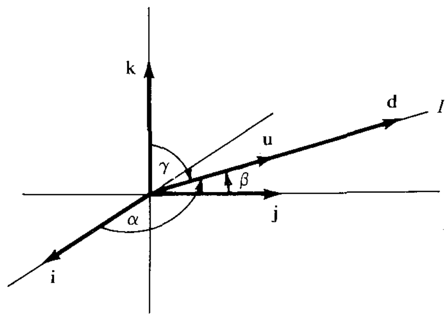


Figure 13.4.9. The direction angles of the line l are α, β, and γ.

 (a) Show that $P_0 + s(\mu, \lambda, \nu)$ gives the same line. (What values of s and t correspond to the same points on the line?)
 (b) Show that $\cos\alpha = \mathbf{i} \cdot \mathbf{u} = \mu$; $\cos\beta = \mathbf{j} \cdot \mathbf{u} = \lambda$; $\cos\gamma = \mathbf{k} \cdot \mathbf{u} = \nu$.
 (c) Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
 (d) Determine the direction angles and cosines of each of the lines in Exercise 54.
 (e) Which lines through the origin have direction angles $\alpha = \beta = \gamma$?

★58. Imagine that you look to the side as you walk in the rain. Now you stop walking.
 (a) How does the (apparent) direction of the falling rain change? (The rain may be falling at an angle because of wind.)
 (b) Explain the observation in (a) in terms of vectors.
 (c) Suppose that there is no wind and that you know your walking speed. How could you measure the speed at which the rain is falling?
 (d) Do part (c) if the rain is falling at an angle.

13.5 The Cross Product

The cross product of two vectors in space is a new vector that is perpendicular to the first two.

What is the velocity of a point on a rotating object? Let $\mathbf{v}_1$ be a vector which points in the direction of the axis of rotation and whose length equals the rotation rate (in radians per unit time). Let $\mathbf{v}_2$ be a vector from a point O on the axis of rotation to a point P on the object (see Fig. 13.5.1). A little thought shows that the velocity $\mathbf{v}$ of the point P has the following properties:

1. $\|\mathbf{v}\| = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\sin\theta$, where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.
2. If $\theta \neq 0$ (so that $\mathbf{v} \neq \mathbf{0}$), $\mathbf{v}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, and the triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v})$ of vectors obeys the right-hand rule (see Fig. 13.5.2).

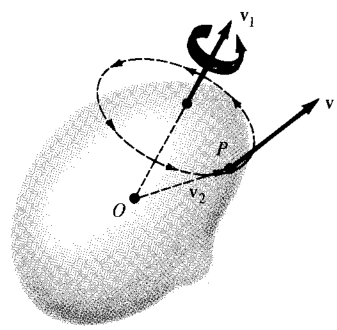
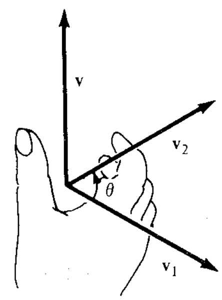


Figure 13.5.1. The point P has velocity vector $\mathbf{v}$.

Figure 13.5.2. Right-hand rule: Place the palm of your hand so that your fingers curl from $\mathbf{v}_1$ in the direction of $\mathbf{v}_2$ through the angle θ. Then your thumb points in the direction of $\mathbf{v}$.



Note that condition 1 says the magnitude of the velocity of P is proportional to the product of the magnitude of the rotation rate and the distance of P from the axis of rotation; furthermore, for fixed $\|\mathbf{v}_2\|$, the velocity is greatest when $\mathbf{v}_2$ is perpendicular to the axis.

Conditions 1 and 2 determine $\mathbf{v}$ uniquely in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$; $\mathbf{v}$ is called the *cross product* (or *vector product*) of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is denoted by $\mathbf{v}_1 \times \mathbf{v}_2$.

We will now determine some properties of the cross product operation. Our ultimate goal is to find a formula for the components of $\mathbf{v}_1 \times \mathbf{v}_2$ in terms of the components of $\mathbf{v}_1$ and $\mathbf{v}_2$. Let us first show that $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is equal to the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Drop the perpendicular PS as shown in Fig. 13.5.3. Then $A = |OR|\,|PS| = |OR|\,|OP|\sin\theta = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\sin\theta = \|\mathbf{v}_1 \times \mathbf{v}_2\|$ by condition 1, proving our claim.

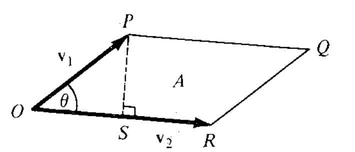


Figure 13.5.3. Calculating the area of $ORQP$.

Example 1 Find all the cross products between the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution We observe first that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$, because the angle between any vector and itself is zero, and $\sin 0 = 0$. Next we observe that $\mathbf{i}\times\mathbf{j}$ must be a multiple of $\mathbf{k}$, since it is perpendicular to $\mathbf{i}$ and $\mathbf{j}$. On the other hand, $\|\mathbf{i}\times\mathbf{j}\| = \|\mathbf{i}\|\,\|\mathbf{j}\|\sin 90° = 1$, so $\mathbf{i}\times\mathbf{j}$ must be $\mathbf{k}$ or $-\mathbf{k}$. The right-hand rule then

shows that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (see Fig. 13.5.4). Next, $\mathbf{j} \times \mathbf{i}$ must be $\mathbf{k}$ or $-\mathbf{k}$; this time the right-hand rule gives $-\mathbf{k}$ as the answer. Similarly, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. ▲

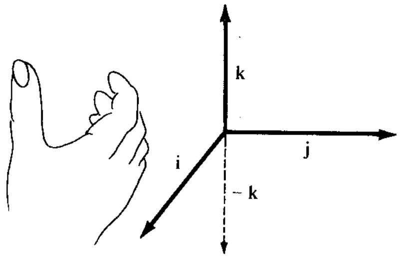


Figure 13.5.4. The right-hand rule requires that $\mathbf{i} \times \mathbf{j}$ equal $\mathbf{k}$, not $-\mathbf{k}$.

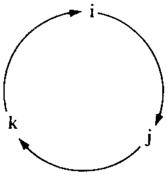


Figure 13.5.5. As we go around the circle, the cross product of any two consecutive vectors equals the third vector. Going backwards produces the negative of the preceding vector.

A good way to remember these products is to write **i**, **j**, and **k** in a circle as in Fig. 13.5.5.

We will now obtain a general formula for the cross product

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}).$$

If we assumed that the usual rules of algebra apply to the cross product, we could use the result of Example 1 to make the following calculation:

$$\begin{aligned}(a_1\mathbf{i} &+ b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})\\ &= a_1\mathbf{i} \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) + b_1\mathbf{j} \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) + c_1\mathbf{k} \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}\\ &= a_1b_2\mathbf{k} + a_1c_2(-\mathbf{j}) + b_1a_2(-\mathbf{k}) + b_1c_2\mathbf{i} + c_1a_2\mathbf{j} + c_1b_2(-\mathbf{i}).\end{aligned}$$

Collecting terms, we have

$$\begin{aligned}(a_1\mathbf{i} &+ b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})\\ &= (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}. \qquad (1)\end{aligned}$$

Although to derive (1) we made the unjustified assumption that some laws of algebra hold for the cross product, it turns out that the result is correct. To see this, we shall show that $\mathbf{u} = (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}$ is indeed the cross product of $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. We accomplish this by verifying that **u** satisfies conditions 1 and 2 in the definition of the cross product.

First, we consider the squared length of **u**:

$$\begin{aligned}&(b_1c_2 - c_1b_2)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - b_1a_2)^2\\ &= b_1^2c_2^2 - 2b_1c_1b_2c_2 + c_1^2b_2^2 + c_1^2a_2^2 - 2a_1c_1a_2c_2 + a_1^2c_2^2 + a_1^2b_2^2 - 2a_1b_1a_2b_2 + b_1^2a_2^2\end{aligned}$$

Now we compute the square of $\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\sin\theta$:

$$\begin{aligned}\|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2\sin^2\theta &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2(1 - \cos^2\theta)\\ &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2 - (\|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\cos\theta)^2\\ &= \|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2\\ &= (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2\end{aligned}$$

which, when it is multiplied out and terms are collected, is the same as $\|\mathbf{u}\|^2$, so $\|\mathbf{u}\| = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\sin\theta$.

Next we check that **u** is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$. We have

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v}_1 &= (b_1c_2 - c_1b_2)a_1 + (c_1a_2 - a_1c_2)b_1 + (a_1b_2 - b_1a_2)c_1\\ &= b_1c_2a_1 - c_1b_2a_1 + c_1a_2b_1 - a_1c_2b_1 + a_1b_2c_1 - b_1a_2c_1\\ &= 0,\end{aligned}$$

since the terms cancel in pairs. Similarly, $\mathbf{u} \cdot \mathbf{v}_2 = 0$, so $\mathbf{u}$ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$.

To check the right-hand rule would require a precise mathematical definition of that rule, which we will not attempt to give. Instead, we will merely remark that the rule is satisfied for all products of standard basis vectors; see Example 1.

We have now shown that the vector $\mathbf{u}$ on the right-hand side of (1) satisfies the conditions in the definition of $\mathbf{v}_1 \times \mathbf{v}_2$, so it must be $\mathbf{v}_1 \times \mathbf{v}_2$. The algebraic rules in the following display may then be verified as a consequence of formula (1) (see Example 7).

The Cross Product

Geometric definition: $\mathbf{v}_1 \times \mathbf{v}_2$ is the vector such that:

1. $\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \sin\theta$, the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ (θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$; $0 \leqslant \theta \leqslant \pi$).
2. $\mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$, and the triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 \times \mathbf{v}_2)$ obeys the right-hand rule.

Component formula:

$$
\begin{aligned}
&(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) \\
&\qquad = (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}
\end{aligned}
$$

Algebraic rules:

1. $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$ if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel or $\mathbf{v}_1$ or $\mathbf{v}_2$ is zero.
2. $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$.
3. $\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3$.
4. $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_3 + \mathbf{v}_2 \times \mathbf{v}_3$.
5. $(a\mathbf{v}_1) \times \mathbf{v}_2 = a(\mathbf{v}_1 \times \mathbf{v}_2)$.

Multiplication table (see Fig. 13.5.5):

		Second Factor		
	$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
First Factor	$\mathbf{i}$	$\mathbf{0}$	$\mathbf{k}$	$-\mathbf{j}$
	$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{0}$	$\mathbf{i}$
	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

Example 2 (a) Compute $(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (\mathbf{j} - \mathbf{k})$. (b) Find $\mathbf{i} \times (\mathbf{i} \times \mathbf{j})$ and $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. Are they equal?

Solution (a) We use the products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, etc. and the algebraic rules as follows:

$$
\begin{aligned}
(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (\mathbf{j} - \mathbf{k}) &= (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times \mathbf{j} - (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times \mathbf{k} \\
&= 3\mathbf{i} \times \mathbf{j} + 2\mathbf{j} \times \mathbf{j} - \mathbf{k} \times \mathbf{j} - 3\mathbf{i} \times \mathbf{k} - 2\mathbf{j} \times \mathbf{k} + \mathbf{k} \times \mathbf{k} \\
&= 3\mathbf{k} + \mathbf{0} + \mathbf{i} + 3\mathbf{j} - 2\mathbf{i} + \mathbf{0} \\
&= -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}.
\end{aligned}
$$

This can be checked using the component formula.

(b) We find that $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$, while $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$, so the two expressions are *not* equal. This example means that the cross product is *not associative*—one cannot move parentheses as in ordinary multiplication. ▲

Example 3 Find the area of the parallelogram spanned by the vectors $\mathbf{v}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v}_2 = -\mathbf{i} - \mathbf{k}$.

Solution We calculate the cross product of $\mathbf{v}_1$ and $\mathbf{v}_2$ by applying the component formula, with $a_1 = 1$, $b_1 = 2$, $c_1 = 3$, $a_2 = -1$, $b_2 = 0$, $c_2 = -1$:

$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= [(2)(-1) - (3)(0)]\mathbf{i} + [(3)(-1) - (1)(-1)]\mathbf{j} + [(1)(0) - (2)(-1)]\mathbf{k}\\ &= -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

Thus the area is

$$\begin{aligned}\|\mathbf{v}_1 \times \mathbf{v}_2\| &= \sqrt{(-2)^2 + (-2)^2 + (2)^2}\\ &= 2\sqrt{3}\,. \ \blacktriangle\end{aligned}$$

Comparing the methods of Examples 2 and 3 shows that it is often easier to use the algebraic rules and the multiplication table directly, rather than using the component formula.

Example 4 Find a unit vector which is orthogonal to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

Solution A vector perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$ is the vector

$$\begin{aligned}(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) &= \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{j} + \mathbf{j} \times \mathbf{k}\\ &= \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i}\\ &= \mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

Since $\|\mathbf{i} - \mathbf{j} + \mathbf{k}\| = \sqrt{3}$, the vector $(1/\sqrt{3})(\mathbf{i} - \mathbf{j} + \mathbf{k})$ is a unit vector perpendicular to $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$. ▲

Example 5 Use the cross product to find the equation of the plane containing the points $(1,1,1)$, $(2,0,0)$, and $(1,1,0)$. (Compare Example 8 of Section 13.4.)

Solution The normal to the plane is perpendicular to any vector which joins two points in the plane, so it is perpendicular to $\mathbf{v}_1 = (1,1,1) - (2,0,0) = (-1,1,1) = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v}_2 = (1,1,1) - (1,1,0) = (0,0,1) = \mathbf{k}$. A vector perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\mathbf{v}_1 \times \mathbf{v}_2 = (-\mathbf{i} + \mathbf{j} + \mathbf{k}) \times \mathbf{k} = -\mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{k} + \mathbf{k} \times \mathbf{k} = \mathbf{j} + \mathbf{i} + \mathbf{0} = \mathbf{i} + \mathbf{j}$, so the equation of the plane has the form $x + y + D = 0$. Since $(1,1,1)$ lies in the plane, $2 + D = 0$, and the equation is $x + y - 2 = 0$. (In Example 8 of Section 13.4, we obtained this result by solving a system of simultaneous equations. Here the cross product does the solving for us.) ▲

Example 6 Find the area of the triangle with vertices $P_1 = (1,1,2)$, $P_2 = (2,-1,0)$, and $P_3 = (1,-1,3)$.

Solution The area of a triangle is half that of the parallelogram spanned by two of its sides. As sides we take the vectors $\mathbf{v}_1 = \overrightarrow{P_1P_2} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v}_2 = \overrightarrow{P_1P_3} = -2\mathbf{j} + \mathbf{k}$. Then

$$\mathbf{v}_1 \times \mathbf{v}_2 = (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \times (-2\mathbf{j} + \mathbf{k}) = -6\mathbf{i} - \mathbf{j} - 2\mathbf{k}.$$

The area of the triangle is thus

$$\begin{aligned}\tfrac{1}{2}\|\mathbf{v}_1 \times \mathbf{v}_2\| &= \tfrac{1}{2}(6^2 + 1^2 + 2^2)^{1/2}\\ &= \tfrac{1}{2}\sqrt{41} \approx 3.20. \ \blacktriangle\end{aligned}$$

Example 7 Prove the algebraic rule 3 by using the component formula.

Solution Let $\mathbf{v}_i = a_i\mathbf{i} + b_i\mathbf{j} + c_i\mathbf{k}$ $(i = 1, 2, 3)$. Then

$$\mathbf{v}_2 + \mathbf{v}_3 = (a_2 + a_3)\mathbf{i} + (b_2 + b_3)\mathbf{j} + (c_2 + c_3)\mathbf{k}$$

and

$$\begin{aligned}\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_3) &= \left[b_1(c_2 + c_3) - c_1(b_2 + b_3)\right]\mathbf{i} \\ &\quad + \left[c_1(a_2 + a_3) - a_1(c_2 + c_3)\right]\mathbf{j} + \left[a_1(b_2 + b_3) - b_1(a_2 + a_3)\right]\mathbf{k} \\ &= (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k} \\ &\quad + (b_1c_3 - c_1b_3)\mathbf{i} + (c_1a_3 - a_1c_3)\mathbf{j} + (a_1b_3 - a_3b_1)\mathbf{k} \\ &= \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_3 . \ \blacktriangle\end{aligned}$$

Example 8 Show that $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$.

Solution Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta \quad \text{and} \quad \mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Squaring both equations and summing gives

$$\begin{aligned}\|\mathbf{u} \times \mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(\sin^2\theta + \cos^2\theta) \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2,\end{aligned}$$

so $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$. ▲

Exercises for Section 13.5

Calculate the cross products in Exercises 1–10.

1. $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{j} - \mathbf{k})$.
2. $\mathbf{i} \times (\mathbf{j} - \mathbf{k})$.
3. $(\mathbf{i} + \mathbf{j}) \times [(\mathbf{k} - \mathbf{j}) + (3\mathbf{j} - 2\mathbf{i} + \mathbf{k})]$.
4. $(a\mathbf{i} + \mathbf{j} - \mathbf{k}) \times \mathbf{i}$.
5. $[(3\mathbf{i} + 2\mathbf{j}) \times 3\mathbf{j}] \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$.
6. $(\mathbf{i} \times \mathbf{j}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.
7. $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + 3\mathbf{k})$.
8. $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{k})$.
9. $(\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.
10. $(3\mathbf{i} - 2\mathbf{k}) \times (3\mathbf{i} - \mathbf{j} - \mathbf{k})$.

Find the area of the parallelogram spanned by the vectors in Exercises 11–14.

11. $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
12. $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.
13. $\mathbf{i}$ and $\mathbf{i} - 2\mathbf{j}$.
14. $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Find a unit vector orthogonal to the pairs of vectors in Exercises 15–18.

15. $\mathbf{i}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
16. $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.
17. $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
18. $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - \mathbf{j}$.

19. Find a unit vector perpendicular to $\mathbf{i} + \mathbf{j}$ and to $\mathbf{i} - \mathbf{j} - \frac{1}{2}\mathbf{k}$ and with a positive $\mathbf{k}$ component.
20. Find a unit vector perpendicular to $\mathbf{i} - \mathbf{j}$ and to $\mathbf{i} + \mathbf{k}$ and with a positive $\mathbf{k}$ component.
21. Find the equation of the plane passing through the points $(0, 0, 0)$, $(2, 0, -1)$, and $(0, 4, -3)$.
22. Find the equation of the plane through the points $(1, 2, 0)$, $(0, 1, -2)$, and $(4, 0, 1)$.
23. Find the equation of the plane through $(1, 1, 1)$ and containing the line which is the intersection of the planes $x - y = 2$ and $y - z = 1$.
24. Find the equation of the plane through the point $(2, 1, 1)$ and containing the line $x = t - 1$, $y = 2t + 1$, $z = -t - 1$.
25. Find the area of the triangle whose vertices are $(0, 1, 2)$, $(3, 4, 5)$, and $(-1, -1, 0)$.
26. Find the area of the triangle whose vertices are $(0, 1, 2)$, $(1, 1, 1)$, and $(2, 1, 0)$.
27. Find the area of the triangle whose vertices are $(0, 0, 0)$, $(0, -1, 1)$, and $(0, 1, -1)$.
28. Find the area of the triangle whose vertices are $(-1, -1, -1)$, $(-1, 0, 1)$, and $(1, 0, -1)$.
29. Prove algebraic rule 5 by using the component formula.
30. Prove the formula in Example 8 by using the component formula.
31. By using the cross product of the vectors $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\cos\psi\mathbf{i} + \sin\psi\mathbf{j}$, verify that $\sin(\theta - \psi) = \sin\theta\cos\psi - \cos\theta\sin\psi$.

32. Let ℓ be a line through a point P_0 in direction $\mathbf{d}$. Show that the distance from a point P to ℓ is given by $\|(\overrightarrow{P_0P}) \times \mathbf{d}\|/\|\mathbf{d}\|$.
33. Let a line in the plane be given by the equation $ax + by = c$. Use the cross product to show that the distance from a point $P = (x, y)$ to this line is given by
$$\frac{|ax + by - c|}{\sqrt{a^2 + b^2}}.$$
34. Use the cross product to find a solution of the following simultaneous equations: $x + y = 0$ and $x - y - 2z = 0$.
35. In mechanics, the *moment M of a force* $\mathbf{F}$ *about a point O* is defined to be the magnitude of $\mathbf{F}$ times the perpendicular distance $\mathbf{d}$ from O to the line of action of $\mathbf{F}$. The *vector moment* $\mathbf{M}$ is the vector of magnitude M whose direction is perpendicular to the plane of O and $\mathbf{F}$, determined by the right-hand rule. Show that $\mathbf{M} = \mathbf{R} \times \mathbf{F}$, where $\mathbf{R}$ is any vector from O to the line of action of $\mathbf{F}$. (See Fig. 13.5.6.)

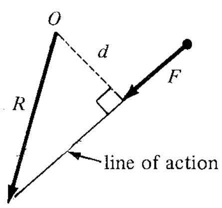


Figure 13.5.6. Moment of a force.

36. The angular velocity $\boldsymbol{\Omega}$ of rotation of a rigid body has direction equal to the axis of rotation and magnitude equal to the angular velocity in radians per second. The sense of $\boldsymbol{\Omega}$ is determined by the right-hand rule.
    (a) Let $\mathbf{r}$ be a vector from the axis to a point P on the rigid body. Show that the quantity $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$ is the velocity of $\mathbf{P}$, as in Fig. 13.5.1, with $\boldsymbol{\Omega} = \mathbf{v}_1$ and $\mathbf{r} = \mathbf{v}_2$.
    (b) Interpret the result for the rotation of a carousel about its axis, with P a point on the circumference.
37. Two media with indices of refraction n_1 and n_2 are separated by a plane surface perpendicular to the unit vector $\mathbf{N}$. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors along the incident and refracted rays, respectively, their directions being those of the light rays. Show that $n_1(\mathbf{N} \times \mathbf{a}) = n_2(\mathbf{N} \times \mathbf{b})$ by using *Snell's law* $\sin\theta_1/\sin\theta_2 = n_2/n_1$, where θ_1 and θ_2 are the angles of incidence and refraction, respectively. (See Fig. 13.5.7.)

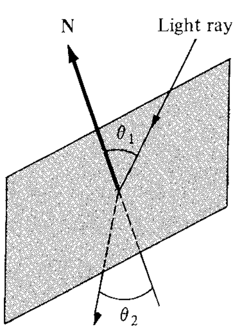


Figure 13.5.7. Snell's law.

★38. Prove the following:
    (a) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{u} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{b}) - (\mathbf{v} \cdot \mathbf{a})(\mathbf{u} \cdot \mathbf{b})$.
    (b) *The Jacobi identity*:
$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}.$$
★39. (a) Using vector methods, show that the distance between two nonparallel lines l_1 and l_2 is given by
$$d = \frac{|(\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{a}_1 \times \mathbf{a}_2)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|},$$
where $\mathbf{v}_1$, $\mathbf{v}_2$ are vectors from the origin to points on l_1 and l_2, respectively, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the directions of l_1 and l_2. [*Hint:* Consider the plane through l_2 which is parallel to l_1. Show that $(\mathbf{a}_1 \times \mathbf{a}_2)/\|\mathbf{a}_1 \times \mathbf{a}_2\|$ is a unit normal for this plane; now project $\mathbf{v}_2 - \mathbf{v}_1$ onto this normal direction.]
    (b) Find the distance between the line l_1 determined by the two points $(-1, -1, 1)$ and $(0, 0, 0)$ and the line l_2 determined by the points $(0, -2, 0)$ and $(2, 0, 5)$.
★40. Use properties of the cross product to explain why, in the discussion of rotation at the beginning of this section, the resulting vector $\mathbf{v}$ does not depend on the choice of the origin O; i.e., what happens if O is replaced by another point O' on the axis of rotation?
★41. When a gyroscope rotating about an axis $\boldsymbol{\Omega}$, as in Fig. 13.5.8, is subject to a force $\mathbf{F}$, the gyroscope responds by moving in the direction $\boldsymbol{\Omega} \times \mathbf{F}$.[5] Show that this fact is consistent with the gyroscopic precession you actually observe in toy gyroscopes.

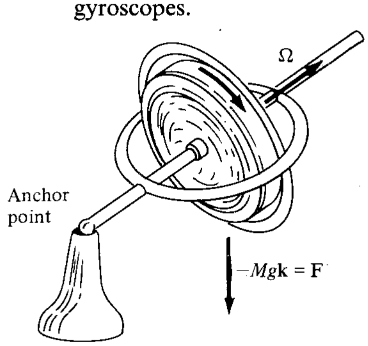


Figure 13.5.8. Gyroscope and cross products.

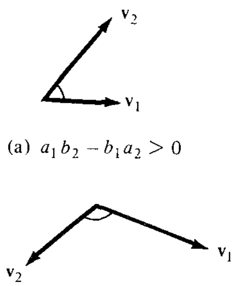
[5] This relationship is easiest to see in an orbiting earth satellite, where the effects of gravity do not complicate the issue. Indeed, the Skylab astronauts had great fun carrying out such experiments. (See Henry S. Cooper, Jr., *A House in Space*, Holt, Rinehart, and Winston, 1976).

13.6 Matrices and Determinants

The cross product can be expressed as a 3×3 *determinant.*

From the point of view of geometry and vectors, we will consider, in turn, 2×2 determinants and matrices, and then the 3×3 case.

If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j}$ are vectors in the plane, to compute the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, we may consider them as vectors in space (with $c_1 = c_2 = 0$) and take the cross-product $\mathbf{v}_1 \times \mathbf{v}_2 = (a_1b_2 - b_1a_2)\mathbf{k}$. The area is then $\|\mathbf{v}_1 \times \mathbf{v}_2\| = |a_1b_2 - b_1a_2|$, but the sign of $a_1b_2 - b_1a_2$ also gives us some information: it is positive if the sense of (shortest) rotation from $\mathbf{v}_1$ to $\mathbf{v}_2$ is counterclockwise and negative if the sense of rotation is clockwise (see Fig. 13.6.1). We may say that the sign of $a_1b_2 - b_1a_2$ determines the *orientation* of the ordered pair of vectors $(\mathbf{v}_1, \mathbf{v}_2)$.



(a) $a_1b_2 - b_1a_2 > 0$

(b) $a_1b_2 - b_1a_2 < 0$

Figure 13.6.1. The sign of $a_1b_2 - b_1a_2$ determines the orientation of the ordered pair $(a_1\mathbf{i} + b_1\mathbf{j}, a_2\mathbf{i} + b_2\mathbf{j})$.

The combination $a_1b_2 - b_1a_2$ of the four numbers a_1, b_1, a_2, and b_2 is denoted by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and is called the *determinant* of these four numbers.

2×2 Determinants

If a, b, c, d are any four numbers, we write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The absolute value of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

equals the area of the parallelogram spanned by the vectors $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{w} = c\mathbf{i} + d\mathbf{j}$. The sign of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

gives the orientation of the pair $(\mathbf{v}, \mathbf{w})$.

Example 1 (a) Find the determinant $\begin{vmatrix} -\pi & 3 \\ \pi/2 & 6 \end{vmatrix}$.

(b) Show that $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix}$. What is their common value?

(c) Prove that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$ (the two columns are interchanged).

Solution (a)

$$\begin{vmatrix} -\pi & 3 \\ \pi/2 & 6 \end{vmatrix} = -6\pi - \frac{3\pi}{2} = -\frac{15\pi}{2}.$$

(b) $$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2;$$
$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = (1)(6) - (2)(4) = -2;$$
$$\begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} = (1)(6) - (4)(2) = -2.$$

Their common value is -2.

(c) $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \text{ and}$$
$$-\begin{vmatrix} b & a \\ d & c \end{vmatrix} = -(bc - ad) = ad - bc. \blacktriangle$$

From the previous section we recall that
$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$$
$$= (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k}.$$

All the components on the right-hand side are determinants. We have:
$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$$
$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{i} + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\mathbf{k}. \tag{1}$$

The middle term can also be written
$$-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\mathbf{j}.$$

Shortly we shall see how to write the cross product in terms of a single 3×3 determinant.

Sometimes we wish to refer to the array
$$\begin{matrix} a & b \\ c & d \end{matrix}$$
of numbers without taking the combination $ad - bc$. In this case, we use the notation
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
and refer to this object as a 2×2 *matrix*. Two matrices are considered equal only when all their corresponding entries are equal; thus, in contrast to the equalities in Example 1(b), the matrices
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$$
are all different. The determinant
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
is a *single* number obtained by combining the *four* numbers in the matrix
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Geometrically, we may think of a matrix as representing a parallelogram (that is, a geometric figure), while the determinant represents only the area of the parallelogram (that is, a number).

Example 2 The transpose of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ obtained by reflection across the main (upper left to lower right) diagonal.

(a) Find the transpose of $\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}$.

(b) Show that the determinant of a matrix is equal to the determinant of its transpose: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

(c) Check (b) for the matrix in (a).

Solution (a) The transpose of $\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -3 \\ 5 & 2 \end{bmatrix}$.

(b) $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

(c) $$\begin{vmatrix} 1 & 5 \\ -3 & 2 \end{vmatrix} = 2 + 15 = 17;$$

$$\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = 2 + 15 = 17. \blacktriangle$$

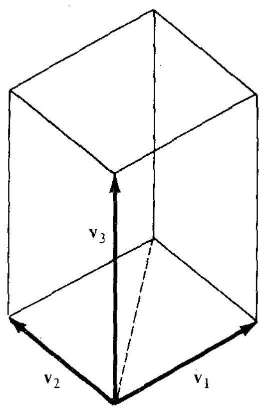


Figure 13.6.2. The absolute value of a determinant equals the volume of the parallelepiped spanned by the rows.

A 3×3 *matrix* consists of nine numbers in a square array, such as

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 9 \\ 3 & 1 & -5 \end{bmatrix}.$$

To define the determinant $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ of a 3×3 matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, we proceed by analogy with the 2×2 case. The rows of the matrix give us three vectors in space:

$$\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k},$$
$$\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k},$$
$$\mathbf{v}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}.$$

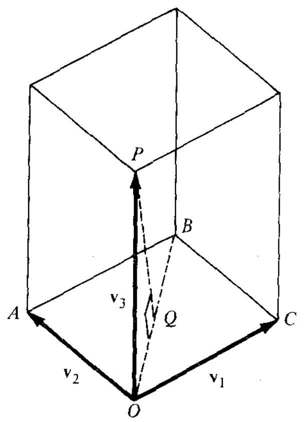


Figure 13.6.3. The volume of the parallelepiped is its height $|PQ|$ times the area of its base.

Since the determinant of a 2×2 matrix represents an area, we should define the determinant in such a way that its absolute value is the *volume* of the parallelepiped spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (See Fig. 13.6.2.)

To compute this volume in terms of the nine entries in the matrix, we drop a perpendicular PQ from the tip of $\mathbf{v}_3$ to the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. It is a theorem of elementary solid geometry that the volume of the parallelepiped is equal to the length of PQ times the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ (see Fig. 13.6.3).

Looking at the right angle OQP, we find that the length of $\overrightarrow{PQ}$ is $\|\mathbf{v}_3\| \, |\cos\theta|$, where θ is the angle between $\mathbf{v}_3$ and $\overrightarrow{PQ}$. On the other hand, PQ is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$, so it is parallel to $\mathbf{v}_1 \times \mathbf{v}_2$, so θ is also the angle between $\mathbf{v}_3$ and $\mathbf{v}_1 \times \mathbf{v}_2$. Now we have:

$$\begin{aligned}\text{Volume} &= (\text{area of base})(\text{height})\\ &= \|\mathbf{v}_1 \times \mathbf{v}_2\|\,\|\mathbf{v}_3\|\,|\cos\theta|\\ &= |\,\|\mathbf{v}_1 \times \mathbf{v}_2\|\,\|\mathbf{v}_3\|\cos\theta|\\ &= |(\mathbf{v}_1 \times \mathbf{v}_2)\cdot\mathbf{v}_3|.\end{aligned}$$

Since the volume is to be the absolute value of the determinant, we define the determinant to be the expression inside the bars:

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \left[(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})\right]\cdot(a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}).$$

By using the component formula for the cross product in equation (1), we find

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}a_3 + \begin{vmatrix} c_1 & a_1\\ c_2 & a_2\end{vmatrix}b_3 + \begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}c_3$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}a_3 - \begin{vmatrix} a_1 & c_1\\ a_2 & c_2\end{vmatrix}b_3 + \begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}c_3. \tag{2}$$

We take equation (2) as the definition of the determinant of a 3×3 matrix; the volume of the parallelepiped spanned by three vectors is thus equal to the absolute value of the determinant of the matrix whose rows are the components of the vectors. The sign of the determinant is interpreted geometrically in Example 9.

Example 3 Evaluate the determinant $\begin{vmatrix} 0 & 0 & 4\\ 2 & -1 & 6\\ 3 & 1 & 2\end{vmatrix}$.

Solution By equation (2),

$$\begin{aligned}\begin{vmatrix} 0 & 0 & 4\\ 2 & -1 & 6\\ 3 & 1 & 2\end{vmatrix} &= \begin{vmatrix} 0 & 4\\ -1 & 6\end{vmatrix}(3) - \begin{vmatrix} 0 & 4\\ 2 & 6\end{vmatrix}(1) + \begin{vmatrix} 0 & 0\\ 2 & -1\end{vmatrix}(2)\\ &= (4)(3) - (-8)(1) + (0)(2)\\ &= 12 + 8 = 20. \blacktriangle\end{aligned}$$

We can express the cross product of two vectors as a single 3×3 determinant. In fact, comparing equation (2), with (1),

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) \times (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ \mathbf{i} & \mathbf{j} & \mathbf{k}\end{vmatrix}. \tag{3}$$

Example 4 Write $(\mathbf{i} + \mathbf{k}) \times (\mathbf{j} - 2\mathbf{k})$ as a determinant.

Solution

$$(\mathbf{i} + \mathbf{k}) \times (\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} 1 & 0 & 1\\ 0 & 1 & -2\\ \mathbf{i} & \mathbf{j} & \mathbf{k}\end{vmatrix}. \blacktriangle$$

Formula (2) is worth memorizing. To do so, notice that the ith entry in the third row is multiplied by the determinant obtained by crossing out the ith column and third row of the original matrix. Such a 2×2 determinant is called a *minor*, and formula (2) is called the *expansion by minors of the third row*.

It turns out that a determinant can be evaluated by expanding in minors of any row or column. (We shall verify this for the second column in Example 7.) To do the expansion, multiply each entry in a given row or column by the 2×2 determinant obtained by crossing out the row and column of the given entry. Signs are assigned to the products according to the checkerboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

(Remember the plus sign in the upper left-hand corner, and you can always reconstruct this pattern.) Thus, the cross product (3) can also be written

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Example 5 (a) Evaluate the determinant

$$\begin{vmatrix} 0 & 0 & 4 \\ 2 & -1 & 6 \\ 3 & 1 & 2 \end{vmatrix}$$

of Example 3 by expanding in minors of the first row.
(b) Find $(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ using a 3×3 determinant.

Solution (a)

$$\begin{vmatrix} 0 & 0 & 4 \\ 2 & -1 & 6 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix}(0) - \begin{vmatrix} 2 & 6 \\ 3 & 2 \end{vmatrix}(0) + \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} 4$$
$$= 0 - 0 + (5)(4) = 20.$$

Since we do not need to evaluate the minors of the zero entries, the expansion by minors of the first row results in a simpler calculation (for this particular matrix) than the expansion by minors of the third row.

(b) $$(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -1 & -3 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & 1 \\ -1 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$
$$= 4\mathbf{i} + 7\mathbf{j} - \mathbf{k}. \blacktriangle$$

Example 6 Find the volume of the parallelepiped spanned by the following vectors: $\mathbf{i} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and $5\mathbf{i} + 4\mathbf{k}$.

Solution The volume is the absolute value of

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 5 & 0 & 4 \end{vmatrix}.$$

If we expand this by minors of the second column, the only nonzero term is

$$\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix}(1) = -11,$$

so the volume equals 11. ▲

To prove that the expansions by minors of rows or columns all give the same result, it is sufficient to compare these expansions for the "general" 3×3 determinant.

Example 7 Show that expanding the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by minors of the second column gives the correct result.

Solution The expansion by minors of the second column is

$$-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} b_1 + \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} b_2 - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} b_3 .$$

Expanding the 2×2 determinants gives

$$\begin{aligned} &-(a_2c_3 - c_2a_3)b_1 + (a_1c_3 - c_1a_3)b_2 - (a_1c_2 - c_1a_2)b_3 \\ &\qquad = -b_1a_2c_3 + b_1c_2a_3 + a_1b_2c_3 - c_1b_2a_3 - a_1c_2b_3 + c_1a_2b_3 . \end{aligned}$$

Collecting the terms in a_3, b_3, and c_3, we get

$$\begin{aligned} &(b_1c_2 - c_1b_2)a_3 - (a_1c_2 - c_1a_2)b_3 + (a_1b_2 - b_1a_2)c_3 \\ &\quad = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} a_3 - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} b_3 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3 \\ &\quad = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} . \ \blacktriangle \end{aligned}$$

The proof that we get the same result by expanding in any row or column is similar.

If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$, $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, and $\mathbf{v}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$, equation (2) can be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 .$$

This is called the *triple product* of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Example 8 Show that $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = (\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2$. In fact, the numbers 1, 2, 3 can be moved cyclically without changing the value of the triple product:

$$\begin{array}{ccc} \nearrow & 1 & \searrow \\ 3 & \leftarrow & 2 \end{array}$$

Solution The triple product $(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2$ is equal to the determinant

$$\begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} .$$

Expanding this by minors of the first row gives

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} a_3 - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} b_3 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3 .$$

Expanding the determinant for $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ by the *third* row gives the same result. ▲

Example 9 Let $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$, $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, and $\mathbf{v}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$. Show that the triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is right-handed (left-handed) if the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is positive (negative). What does it mean if the determinant is zero?

Solution Let $\mathscr{P}$ be the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. (We assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel; otherwise $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$, and the determinant is zero.) Then $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is right handed (left handed) if and only if $\mathbf{v}_3$ lies on the same (opposite) side of $\mathscr{P}$ as $\mathbf{v}_1 \times \mathbf{v}_2$; that is, if and only if $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ is positive (negative); but $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ is the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} .$$

If the determinant is zero (but $\mathbf{v}_1 \times \mathbf{v}_2$ is not zero), then $\mathbf{v}_3$ must lie *in* the plane $\mathscr{P}$. In general, we may say that the determinant is zero when the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ fail to span a solid parallelepiped, but instead lie in a plane, lie on a line, or are all zero. Such triples of vectors are said to be *linearly dependent* and are neither right-handed nor left-handed. ▲

3 × 3 Determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} a_3 - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} b_3 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3$$

The determinant can be expanded by minors of any row or column. If

$$\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k},$$
$$\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k},$$
$$\mathbf{v}_3 = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k},$$

then the determinant equals $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ which is also called the *triple product* of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

The absolute value of the determinant equals the volume of the parallelepiped spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

The sign of the determinant tells whether the triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is right- or left-handed.

Exercises for Section 13.6

Evaluate the determinants in Exercises 1–10.

1. $\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$
2. $\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$
3. $\begin{vmatrix} 6 & 5 \\ 12 & 10 \end{vmatrix}$
4. $\begin{vmatrix} 0 & 0 \\ 3 & 17 \end{vmatrix}$
5. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$
6. $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$
7. $\begin{vmatrix} 4 & 3 \\ 1 & 7 \end{vmatrix}$
8. $\begin{vmatrix} 1-x & -1 \\ 1 & -1-x \end{vmatrix}$
9. $\begin{vmatrix} a & b \\ 0 & c \end{vmatrix}$
10. $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$

Prove the identities in Exercises 11–14.

11. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ (the two rows are interchanged).
12. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+c & b+d \\ c & d \end{vmatrix}$ (the last row is added to the first).
13. $\begin{vmatrix} ra & rb \\ c & d \end{vmatrix} = r\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (the first row is multiplied by a constant).
14. $\begin{vmatrix} a-sb & b \\ c-sd & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (a constant times the second column is subtracted from the first).

Evaluate the determinants in Exercises 15–24.

15. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$
16. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$
17. $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix}$
18. $\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$
19. $\begin{vmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix}$
20. $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$
21. $\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix}$
22. $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix}$
23. $\begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix}$
24. $\begin{vmatrix} a & 0 & 0 \\ e & b & d \\ 0 & 0 & c \end{vmatrix}$

Write the cross products in Exercises 25–30 as determinants and evaluate.

25. $(3\mathbf{i} - \mathbf{j}) \times (\mathbf{j} + \mathbf{k})$.
26. $(\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.
27. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k})$.
28. $(2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}) \times \mathbf{k}$.
29. $(\mathbf{i} - \mathbf{k}) \times (\mathbf{i} + \mathbf{k})$.
30. $(\mathbf{j} + 2\mathbf{k}) \times (32\mathbf{j} + 64\mathbf{k})$.
31. Find the volume of the parallelepiped spanned by the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$, and $3\mathbf{k}$.
32. Find the volume of the parallelepiped spanned by $\mathbf{i} - \mathbf{j}$, $\mathbf{j} - \mathbf{k}$, and $\mathbf{k} + \mathbf{i}$.
33. Find the volume of the parallelepiped with one vertex at $(1, 1, 2)$ and three adjacent vertices at $(2, 0, 2)$, $(3, 1, 3)$, and $(2, 2, -3)$.
34. Find the other four vertices of the parallelepiped in Exercise 33 and use them to recompute the volume.
35. Check that expanding the determinant
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
by (a) minors of the second row and (b) minors of the third column gives the correct result.
36. Show that if the first two rows of a 3×3 matrix are interchanged, the determinant changes sign.
37. Show that if the first two columns of a matrix are interchanged, the determinant changes sign.
38. Use Exercise 36 to verify that $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = -(\mathbf{v}_2 \times \mathbf{v}_1) \cdot \mathbf{v}_3$.
39. Verify that $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = (\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1$.
40. Verify that $(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_1 + \mathbf{v}_2) = 2\mathbf{v}_1 \times \mathbf{v}_2$.
41. Show by drawing appropriate diagrams that if $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a right-handed triple, then so is $(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1)$.
42. Show that if two rows or columns of a 3×3 matrix are equal, then the determinant of the matrix is zero.
43. Show that if
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$
then the solutions to the equations
$$ax + by = e,$$
$$cx + dy = f$$
are given by the formulas
$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$
This result is called *Cramer's rule*[6] (We have already used it without the language of determinants in our discussion of Wronskians in the Supplement to Section 12.7.)
44. Use Cramer's rule (Exercise 43) to solve the equations $4x + 3y = 2$; $2x - 6y = 1$.

★45. Suppose that the determinant
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

[6] Gabriel Cramer (1704–1752) published this rule in his book, *Introduction à l'analyse des lignes courbes algébriques* (1750). However, it was probably known to Maclaurin in 1729. For systems of n equations in n unknowns, there is a generalization of this rule, but it can be inefficient to use on a computer when n is large. (See Exercises 45 for the case $n = 3$.)

is unequal to zero. Show that the solution of the equations

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3$$

is given by the formulas

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}.$$

This result is called Cramer's rule for 3×3 systems.

46. Use Cramer's rule (Exercise 45) to solve

$$\begin{aligned} -x+y \quad\; &= 14, \\ 2x+y+z &= 8, \\ x+y+5z &= -1. \end{aligned}$$

Use Cramer's rule to solve the systems in Exercises 47 and 48.

47. $2x + 3y = 5$; $3x - 2y = 9$.
48. $x + y + z = 3$; $x - y + z = 4$; $x + y - z = 5$.
49. Check that

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

That is, check that the determinant of the transpose of a 3×3 matrix is equal to the determinant of the original matrix.

★50. Show that adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged; that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 + \lambda a_1 & b_2 + \lambda b_1 & c_2 + \lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

[In fact, adding a multiple of any row (column) of a matrix to another row (column) leaves the determinant unchanged.]

★51. Justify the steps in the following computation:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 10 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} \\ &= \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} = 33 - 36 = -3. \end{aligned}$$

★52. Follow the technique of Exercise 51 to evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ -1 & 2 & 1 \end{vmatrix}.$$

(Add -2 times the first row to the second row, then add the first row to the third row.)

★53. Use the technique in Exercise 51 to evaluate the 3×3 determinants in Exercises 18 and 19.

★54. Show that the plane which passes through the three points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ consists of the points $P = (x, y, z)$ given by

$$\begin{vmatrix} a_1 - x & a_2 - y & a_3 - z \\ b_1 - x & b_2 - y & b_3 - z \\ c_1 - x & c_2 - y & c_3 - z \end{vmatrix} = 0.$$

[*Hint:* Write the determinant as a triple product.]

Review Exercises for Chapter 13

Complete the calculations in Exercises 1–16.

1. $(3,2) + (-1,6) =$
2. $2(1,-4) + (1,5) =$
3. $(1,2,3) + 2(-1,-2,7) =$
4. $2[(-1,0,1) + (6,0,2)] - (0,0,1) =$
5. $(3\mathbf{i} + 2\mathbf{j}) + (8\mathbf{i} - \mathbf{j} - \mathbf{k}) =$
6. $(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - 6(\mathbf{i} - \mathbf{j} - \mathbf{k}) =$
7. $(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} - \mathbf{k}) =$
8. $(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) =$
9. $(8\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} - \mathbf{k}) =$
10. $(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j}) =$
11. $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j}) =$
12. $[(2\mathbf{i} - \mathbf{j}) \times (3\mathbf{i} + \mathbf{j})] \cdot (2\mathbf{j} + \mathbf{k}) =$
13. $\mathbf{u} \times \mathbf{v} = ?$, where $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{k}$.
14. $\mathbf{u} \cdot \mathbf{v} = ?$, where $\mathbf{u} = 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + \mathbf{i}$.
15. $\mathbf{u} - 3\mathbf{v} = ?$, where $\mathbf{u}$ and $\mathbf{v}$ are as in Exercise 13.
16. $\mathbf{u} + 6\mathbf{v} = ?$, where $\mathbf{u}$ and $\mathbf{v}$ are as in Exercise 14.
17. Find a unit vector orthogonal to $3\mathbf{i} + 2\mathbf{k}$ and $\mathbf{j} - \mathbf{k}$.
18. Find a unit vector orthogonal to $\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$.
19. Find the volume of the parallelepiped spanned by $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i}$, and $\mathbf{j} - \mathbf{k}$.
20. Find the volume of the parallelepiped spanned by $\mathbf{i} + \mathbf{j}$, $\mathbf{i} - \mathbf{j}$, and $\mathbf{i} + \mathbf{k}$.
21. (a) Draw the vector $\mathbf{v}$ joining $(-2,0)$ to $(4,6)$ and find the components of $\mathbf{v}$. (b) Add $\mathbf{v}$ to the vector joining $(-2,0)$ to $(1,1)$.
22. Find the intersection of the medians of the triangle with vertices at $(0,0)$, $(1,\frac{1}{2})$, and $(2,0)$.
23. Let PQR be a triangle in the plane. For each side of the triangle, construct the vector perpendicular to that side, pointing into the triangle, and having the same length as the side. Prove that the sum of the three vectors is zero.

24. Show that the diagonals of a rhombus are perpendicular to each other.
25. A bird is headed northeast with speed 40 kilometers per hour. A wind from the north at 15 kilometers per hour begins to blow, but the bird continues to head northeast and flies at the same rate relative to the air. Find the speed of the bird relative to the earth's surface.
26. An airplane flying in a straight line at 500 miles per hour for 12 minutes moves 35 miles north and 93.65 miles east. How much does its altitude change? Can you determine whether the airplane is climbing or descending? (Ignore the curvature of the earth.)
27. The work W done in moving an object from $(0,0)$ to $(7,2)$ subject to a force $\mathbf{F}$ is $W = \mathbf{F}\cdot\mathbf{r}$ where $\mathbf{r}$ is the vector with head at $(7,2)$ and tail at $(0,0)$. The units are feet and pounds.
   (a) Suppose the force $\mathbf{F} = 10\cos\theta\mathbf{i} + 10\sin\theta\mathbf{j}$. Find W in terms of θ.
   (b) Suppose the force $\mathbf{F}$ has magnitude 6 lbs and makes an angle of $\pi/6$ radian with the horizontal, pointing right. Find W in feet-lbs.
28. If a particle with mass m moves with velocity $\mathbf{v}$, its *momentum* is $\mathbf{p} = m\mathbf{v}$. In a game of marbles, a marble with mass 2 grams is shot with velocity 2 meters per second, hits two marbles with mass 1 gram each, and comes to a dead halt. One of the marbles flies off with a velocity of 3 meters per second at an angle of 45° to the incident direction of the larger marble as in Fig. 13.R.1. Assuming that the total momentum before and after the collision is the same (law of conservation of momentum), at what angle and speed does the second marble move?

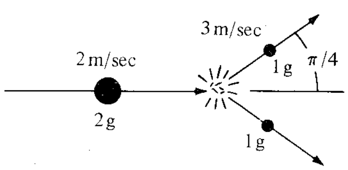


Figure 13.R.1. Momentum and marbles.

Write an equation, or set of equations, to describe each of the following geometric objects in Exercises 29–40.

29. The line through $(1,1,2)$ and $(2,2,3)$.
30. The line through $(0,0,-1)$ and $(1,1,3)$.
31. The line through $(1,1,1)$ in the direction of $\mathbf{i}-\mathbf{j}-\mathbf{k}$.
32. The line through $(1,-1,2)$ in the direction of $\mathbf{i}+\mathbf{j}+3\mathbf{k}$.
33. The plane through $(1,1,2)$, $(2,2,3)$, and $(0,0,0)$.
34. The plane through the points $(1,2,3)$, $(1,-1,1)$, and $(-1,1,1)$.
35. The plane through $(1,1,-1)$ and orthogonal to $\mathbf{i}-\mathbf{j}-\mathbf{k}$.
36. The plane through $(1,-1,6)$ and orthogonal to $\mathbf{i}+\mathbf{j}+\mathbf{k}$.
37. The line perpendicular to the plane in Exercise 33 and passing through $(0,0,3)$.
38. The line perpendicular to the plane in Exercise 34 and passing through $(1,1,1)$.
39. The line perpendicular to the plane in Exercise 35 and passing through $(2,3,1)$.
40. The line perpendicular to the plane in Exercise 36 and passing through the origin.

In Exercises 41–46, find a unit vector which has the given property.

41. Orthogonal to the plane $x-6y+z=12$.
42. Parallel to the line $x=3t+1$, $y=16t-2$, $z=-(t+2)$.
43. Orthogonal to $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and to $\mathbf{k}$.
44. Parallel to both the planes $8x+y+z=1$ and $x-y-z=0$.
45. At an angle of 30° to $\mathbf{i}$ and makes equal angles with $\mathbf{j}$ and $\mathbf{k}$.
46. Orthogonal to the line $x=2t-1$, $y=-t-1$, $z=t+2$, and the vector $\mathbf{i}-\mathbf{j}$.

47. Suppose that $\mathbf{v}$ and $\mathbf{w}$ each are parallel to the (x,y) plane. What can you say about $\mathbf{v}\times\mathbf{w}$?
48. Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three vectors. Explain how to find the angle between $\mathbf{w}$ and the plane determined by $\mathbf{u}$ and $\mathbf{v}$.
49. Describe the set of all lines through the origin in space which make an angle of $\pi/3$ with the x axis.
50. Consider the set of all points P in space such that the vector from O to P has length 2 and makes an angle of 45° with $\mathbf{i}+\mathbf{j}$.
   (a) What kind of geometric object is this set?
   (b) Describe this set using equation(s) in x, y, and z.
51. Let a triangle have adjacent sides $\mathbf{a}$ and $\mathbf{b}$.
   (a) Show that $\mathbf{c}=\mathbf{b}-\mathbf{a}$ is the third side.
   (b) Show that $\mathbf{c}\times\mathbf{a}=\mathbf{c}\times\mathbf{b}$.
   (c) Derive the law of sines (see p. 263).
52. Find the equation of the plane through $(1,2,-1)$ which is parallel to both $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{i}-3\mathbf{k}$.
53. Thales' theorem states that the angle θ in Fig. 13.R.2(a) is $\pi/2$. Prove this using the vectors $\mathbf{a}$ and $\mathbf{b}$ shown in Fig. 13.R.2(b).

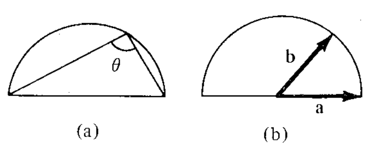


Figure 13.R.2. Triangle inscribed in a semicircle.

54. Show that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices.

Evaluate the determinants in Exercises 55–62.

55. $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$

56. $\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$

57. $\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}$

58. $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$

59. $\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$

60. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

61. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$

62. $\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{vmatrix}$

63. Find the area of the parallelogram spanned by $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $8\mathbf{i} - \mathbf{k}$.
64. Find the area of the parallelogram spanned by $2\mathbf{i} - \mathbf{j}$ and $3\mathbf{i} - 2\mathbf{j}$.
65. Find the volume of the parallelepiped spanned by $\mathbf{i} - \mathbf{j} - \mathbf{k}$, $2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, and $\frac{8}{3}\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$.
66. Find the volume of the parallelepiped spanned by $2\mathbf{j} + \mathbf{i}$, $\mathbf{i} - \mathbf{j}$, and $\mathbf{k}$.
67. The volume of a *tetrahedron* with concurrent edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is given by $V = (1/6)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
    (a) Express the volume as a determinant.
    (b) Evaluate V when $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j}$.
68. A tetrahedron sits in xyz coordinates with one vertex at $(0,0,0)$, and the three edges concurrent at $(0,0,0)$ are coincident with the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
    (a) Draw a figure and label the heads of the vectors as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
    (b) Find the center of mass of each of the four triangular faces of the tetrahedron.
69. Let $\mathbf{r}_1, \dots, \mathbf{r}_n$ be vectors from 0 to the masses $m_1, \dots, m_n$. The *center of mass* is the vector

$$\mathbf{c} = \left(\sum_{i=1}^{n} m_i \mathbf{r}_i\right) \Big/ \left(\sum_{i=1}^{n} m_i\right)$$

Show that for any vector $\mathbf{r}$,

$$\sum_{i=1}^{n} m_i \|\mathbf{r} - \mathbf{r}_i\|^2 = \sum_{i=1}^{n} m_i \|\mathbf{r}_i - \mathbf{c}\|^2 + m\|\mathbf{r} - \mathbf{c}\|^2,$$

where $m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

70. Solve the following equations using Cramer's rule (Exercise 43, Section 13.6): $x + y = 2$; $3x - y = 4$.
71. Solve by using determinants (Exercise 45, Section 13.6): $x - y + 2z = 4$; $3x + y + z = 1$; $4x - y - z = 2$.
72. Use Exercise 50, Section 13.6 to show that

$$\begin{vmatrix} 66 & 628 & 246 \\ 88 & 435 & 24 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 68 & 627 & 247 \\ 86 & 436 & 23 \\ 2 & -1 & 1 \end{vmatrix}.$$

73. Evaluate

$$\begin{vmatrix} 6 & 2 & -3 \\ 2 & 2 & 3 \\ 4 & 8 & -1 \end{vmatrix}$$

using Exercise 50, Section 13.6.

74. Use Exercise 50, Section 13.6 to show that

$$\begin{vmatrix} n & n+1 & n+2 \\ n+3 & n+4 & n+5 \\ n+6 & n+7 & n+8 \end{vmatrix}$$

has the same value no matter what n is. What is this value?

75. Show that for all x, y, z,

$$\begin{vmatrix} x+2 & y & z \\ z & y+1 & 10 \\ 5 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} y & x+2 & z \\ 1 & z-x-2 & 10-z \\ 5 & 5 & 2 \end{vmatrix}$$

76. Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \neq 0$$

if x, y, and z are all different.

77. If the triple product $(\mathbf{v} \times \mathbf{j}) \cdot \mathbf{k}$ is zero, what can you say about the vector $\mathbf{v}$?
78. Suppose that the three vectors $a_i\mathbf{i} + b_i\mathbf{j} + c_i\mathbf{k}$ for $i = 1$, 2, and 3 are unit vectors, each orthogonal to the other two. Find the value of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

79. A sphere of radius 10 centimeters with center at $(0,0,0)$ rotates about the z axis with angular velocity 4 radians per second such that the rotation looks counterclockwise from the positive z axis.
    (a) Find the rotation vector $\mathbf{\Omega}$ (see Section 13.5, Exercise 36).
    (b) Find the velocity $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ when $\mathbf{r} = 5\sqrt{2}(\mathbf{i} - \mathbf{j})$ is on the "equator."
80. A pair of dipoles are located at a distance r from each other. The magnetic potential energy P is given by $P = -\mathbf{m}_1 \cdot \mathbf{B}_2$ (*dipole-dipole interaction potential*), where the first dipole has moment $\mathbf{m}_1$ in the external field $\mathbf{B}_2$ of the second dipole. In MKS units,

$$\mathbf{B}_2 = \mu_0 \frac{-\mathbf{m}_2 + 3(\mathbf{m}_2 \cdot \mathbf{1}_r)\mathbf{1}_r}{4\pi r^3},$$

where $\mathbf{1}_r$ is a unit vector, and μ_0 is a scalar constant.

(a) Show that

$$P = \mu_0 \frac{\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_2 \cdot \mathbf{1}_r)(\mathbf{m}_1 \cdot \mathbf{1}_r)}{4\pi r^3}.$$

(b) Find P when $\mathbf{m}_1$ and $\mathbf{m}_2$ are perpendicular.

81. (a) Suppose that $\mathbf{v} \cdot \mathbf{w} = 0$ for all vectors $\mathbf{w}$. Show that $\mathbf{v} = \mathbf{0}$. [*Note:* This is not the same thing as showing that $\mathbf{0} \cdot \mathbf{w} = 0$.]
(b) Suppose that $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all vectors $\mathbf{w}$. Show that $\mathbf{u} = \mathbf{v}$.
(c) Suppose that $\mathbf{v} \cdot \mathbf{i} = \mathbf{v} \cdot \mathbf{j} = \mathbf{v} \cdot \mathbf{k} = 0$. Show that $\mathbf{v} = \mathbf{0}$.
(d) Suppose that $\mathbf{u} \cdot \mathbf{i} = \mathbf{v} \cdot \mathbf{i}$, $\mathbf{u} \cdot \mathbf{j} = \mathbf{v} \cdot \mathbf{j}$, and $\mathbf{u} \cdot \mathbf{k} = \mathbf{v} \cdot \mathbf{k}$. Show that $\mathbf{u} = \mathbf{v}$.

82. Let A, B, C, D be four points in space. Consider the tetrahedron bounded by the four triangles $\triangle_1 = BCD$, $\triangle_2 = ACD$, $\triangle_3 = ABD$, and $\triangle_4 = ABC$. The triangle $\triangle_i$ is called the *ith face of the tetrahedron*. For each i, there is a unique vector $\mathbf{v}_i$ defined as follows: $\mathbf{v}_i$ is perpendicular to the face $\triangle_i$ and points *into* the tetrahedron; the length of $\mathbf{v}_i$ is equal to the area of $\triangle_i$.
(a) Prove that for any tetrahedron $ABCD$, the sum $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_2 + \mathbf{v}_4$ is zero. [*Hint:* Use algebraic properties of the cross-product.]
★(b) Try to generalize the result of part (a) to more complicated polyhedra. (A polyhedron is a solid which is bounded by planar figures.) In other words, show that the sum of the inward normals, with lengths equal to areas of the sides, is zero. You may want to do some numerical calculations if you cannot prove anything, or you may want to restrict yourself to a special class of figures (distorted cubes, decapitated tetrahedra, figures with five vertices, and so forth).
★(c) There is a physical interpretation to the results in parts (a) and (b). If the polyhedron is immersed in a fluid under *constant* pressure p, then the force acting on the ith face is $p\mathbf{v}_i$. Interpret the result $\sum \mathbf{v}_i = 0$ in this context. Does this contradict the fact that water pressure tends to buoy up an immersed object? [*Note:* There is a version of all this material for smooth surfaces. It is related to a result called the *divergence theorem* and involves partial differentiation and surface integrals. See Chapter 18.]

83. Let $P = (1,2)$ and $Q = (2,1)$. Sketch the set of points in the plane of the form $rP + sQ$, where r and s are:
(a) positive integers;
(b) integers;
(c) positive real numbers;
(d) real numbers.

84. Repeat Exercise 83 with $P = (1,2)$ and $Q = (2,4)$.

★85. Repeat Exercise 83 using the points $P = (1,2)$ and $Q = (\pi, 2\pi)$. (You may have to guess parts (a) and (b).)

★86. There are two unit vectors such that if they were drawn on the axes of Fig. 13.R.3, their heads and tails would appear to be at the same point (that is, they would be viewed head on). Approximately what are these vectors? [*Hint:* Suppose that when you tried to plot a point P, the resulting dot on the paper fell right where the axes cross.]

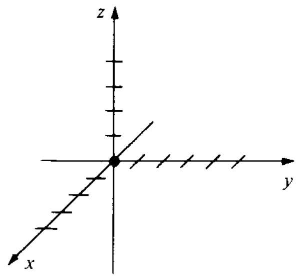


Figure 13.R.3. Which unit vectors would be drawn as the dot at the orign?

★87. A *regular tetrahedron* is a solid bounded by four equilateral triangles. Use vector methods to find the angle between the planes containing two of the faces.

★88. In integrating by partial fractions, we are led to the problem of expressing a rational function

$$\frac{c_2x^2 + c_1x + c_0}{(x - r_1)(x - r_2)(x - r_3)}$$

as a sum

$$\frac{a_2}{x - r_1} + \frac{a_2}{x - r_2} + \frac{a_3}{x - r_3}.$$

(a) Show that the undetermined coefficients a_1, a_2, a_3 satisfy a system of three simultaneous linear equations.
(b) Applying Cramer's rule (Exercise 45, Section 13.6) to this system, show that the determinant D is equal to

$$\begin{vmatrix} 1 & 1 & 1 \\ r_2 + r_3 & r_3 + r_1 & r_1 + r_2 \\ r_2r_3 & r_3r_1 & r_1r_2 \end{vmatrix}.$$

(c) Evaluate the determinant in (b) and show that it is nonzero whenever r_1, r_2, and r_3 are all different.
(d) Conclude that the decomposition into partial fractions is always possible if r_1, r_2, and r_3 are all different.
(e) What happens if $r_1 = r_2$? Give an example.

★89. Read Chapter 5 of Friedrichs' book, *From Pythagoras to Einstein* (Mathematical Association of America, New Mathematical Library, **16** (1965)) on the application of vectors to the study of elastic impacts, and prepare a two-page written report on your findings.